

Equational properties of stratified least fixed points

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Abstract

Recently, a novel fixed point operation has been introduced over certain non-monotonic functions between stratified complete lattices and used to give semantics to logic programs with negation and boolean context-free grammars. We prove that this new operation satisfies ‘the standard’ identities of fixed point operations as described by the axioms of iteration theories. We also study this new fixed point operation in connection with lambda-abstraction.

1 Introduction

The semantics of negation free logic programs is classically defined as the least fixed point of the ‘immediate consequence operation’ canonically associated with the program, cf. [28]. Since this operation is monotonic, the existence of the least fixed point is guaranteed by the well-known Knaster-Tarski theorem [27]. However, for programs with negation, the immediate consequence operation is not necessarily monotonic and fixed points are not guaranteed to exist. The well-founded semantics [21, 23] of logic programs with negation is based on a three-valued (or sometimes four-valued) logic and defines the semantics of a program as the least fixed point of the so-called ‘stable operation’ associated with the program with respect to the information, or knowledge, or Fitting ordering [20]. The well-founded approach has led to the development of a deep abstract fixed point theory for non-monotonic functions which in turn has successfully been applied to problems in various areas beyond logic programming, see [8, 9, 20, 29] for a sampling of articles covering such results.

Another approach to the semantics of logic programs with negation based on an infinite structure of truth values was introduced in [24]. It has been demonstrated that the immediate consequence operation associated with a logic program has a unique minimum model with respect to a novel ordering of the possible interpretations of the program variables over the truth values. An advantage of this approach is that it uses the immediate consequence operation in a direct way. A disadvantage is that it relies on a more complex logic of truth values. However, it does provide more information about the level of certainty of truth or falsity. The development of an abstract fixed point theory underlying the infinite valued approach has recently been undertaken in [18, 19]. In [18], certain stratified complete lattices –called models– were defined, consisting of a complete lattice (L, \leq) and a family $(\sqsubseteq_\alpha)_{\alpha < \kappa}$ of preorderings indexed by the ordinals α less than a fixed nonzero ordinal κ . Several axioms were imposed on models relating the lattice order \leq to the preorderings \sqsubseteq_α . It was established that in such models the preorderings \sqsubseteq_α determine another complete lattice structure (L, \sqsubseteq) , and that if an endofunction of a model satisfies some weak monotonicity or continuity property (it is α -monotonic or α -continuous for each ordinal $\alpha < \kappa$), then it has a least pre-fixed point with respect to the ordering \sqsubseteq , which is a fixed point. (These functions are not necessarily monotonic w.r.t. the ordering \sqsubseteq .) This fixed point theorem has been applied to higher order logic programs and boolean grammars, cf. [6, 19].

A general study of the equational properties of fixed point operations in the context of Lawvere theories or the slightly more general cartesian categories has been provided in [4]. Several other formalisms may

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also be used for the same purpose including abstract clones or μ -expressions, or let-rec expressions. It has been shown that the major fixed point operations commonly used in computer science, including the least fixed point operation over monotonic or continuous functions between complete lattices or cpo's, or in continuous or rational theories [30], the unique fixed point operation over contractive functions of complete metric spaces or in Elgot's iterative theories [10], the initial fixed point operation over continuous functors over certain categories or in algebraically complete categories, and many other models share the equational properties captured by the axioms of iteration theories, or iteration categories, cf. [3, 11, 17] and [4]. For a recent survey, see [16].

Our main aim in this paper is to show that, in conjunction with the cartesian structure, the new fixed point operation introduced in [18] also satisfies the axioms of iteration theories. It then *follows* that an identity is satisfied by the new fixed point operation iff it holds in all iteration theories. We also define cartesian closed categories of models and establish the abstraction identity introduced in [5] that connects the fixed point operation to lambda abstraction.

The paper is organized as follows. In Section 2, we recall the axioms of models from [18]. We point out that models and α -monotonic or α -continuous functions form cartesian categories denoted \mathbf{Mod}_m and \mathbf{Mod}_c . Then, using the fixed point theorem of [18], in Section 3 we endow \mathbf{Mod}_m and \mathbf{Mod}_c with a (parametrized) fixed point operation. Section 4 is devoted to proving that the identities of iteration theories hold in both categories \mathbf{Mod}_m and \mathbf{Mod}_c . In Section 5 we introduce cartesian closed categories of models and establish the abstraction identity that connects the fixed point operation to lambda abstraction. In Section 6 we consider several subcategories of \mathbf{Mod}_m and \mathbf{Mod}_c .

Some notation. Composition of morphisms $f : L \rightarrow L'$ and $g : L' \rightarrow L''$ in any category is denoted $g \circ f$. The identity morphism associated with an object L is denoted \mathbf{id}_L . Function application is written $f(x)$, or just fx .

2 Stratified complete lattices

Let κ be a fixed nonzero ordinal, typically a limit ordinal. In [18], stratified complete lattices equipped with a family of preorderings indexed by the ordinals $\alpha < \kappa$ subject to certain axioms were considered. In this section we recall the definition of these structures. Following [18], we also define α -monotonic and α -continuous functions between models and prove that they yield cartesian categories (cc's). For elementary facts about categories, the reader is referred to [1].

Suppose that (L, \leq) is a complete lattice [7] with least and greatest elements \perp and \top which is equipped with a family $(\sqsubseteq_\alpha)_{\alpha < \kappa}$ of preorderings. For each $\alpha < \kappa$, let $=_\alpha$ denote the equivalence relation determined by \sqsubseteq_α , so that $x =_\alpha y$ iff $x \sqsubseteq_\alpha y$ and $y \sqsubseteq_\alpha x$, for all $x, y \in L$. We say that $(L, \leq, (\sqsubseteq_\alpha)_{\alpha < \kappa})$ is a *stratified complete lattice*, or a *model*, for short, if the following two axioms hold.

Ax1. For all $\alpha < \beta < \kappa$, \sqsubseteq_β is included in $=_\alpha$, i.e., for all $x, y \in L$, if $x \sqsubseteq_\beta y$ then $x =_\alpha y$.

Ax2. The intersection $\bigcap_{\alpha < \kappa} =_\alpha$ of the relations $=_\alpha$ is the equality relation, so that if $x, y \in L$ with $x =_\alpha y$ for all $\alpha < \kappa$, then $x = y$.

Thus, for all $x, y \in L$, we have $x = y$ iff $x =_\alpha y$ for all $\alpha < \kappa$. Moreover, we say that a stratified complete lattice $(L, \leq, (\sqsubseteq_\alpha)_{\alpha < \kappa})$ is a *model*, if it additionally satisfies the following two axioms, where $(x)_\alpha = \{y : \forall \beta < \alpha \ x =_\beta y\}$ for all $x \in L$.

Ax3. For every $\alpha < \kappa$, $x \in L$ and $X \subseteq (x)_\alpha$, there exists some $z \in (x)_\alpha$ with the following properties:

- $X \sqsubseteq_\alpha z$, i.e., $y \sqsubseteq_\alpha z$ for all $y \in X$,
- for all $y \in (x)_\alpha$, if $X \sqsubseteq_\alpha y$ then $z \sqsubseteq_\alpha y$ and $z \leq y$.

Ax4. For every $\alpha < \kappa$ and nonempty $X \subseteq L$ and $y \in L$, if $X =_\alpha y$ (i.e., $x =_\alpha y$ for all $x \in X$), then $\bigvee X =_\alpha y$.

Example 2.1 [18, 24] Consider the following linearly ordered set V of truth values:

$$F_0 < F_1 < \cdots < F_\alpha < \cdots < 0 < \cdots < T_\alpha < \cdots < T_1 < T_0,$$

where $\alpha < \Omega$, the first uncountable ordinal. Let Z denote a nonempty set of (propositional) variables and consider the set $L = V^Z$ equipped with the pointwise ordering, so that for all $f, g \in L$, $f \leq g$ iff $fz \leq gz$ for all $z \in Z$. Then (L, \leq) is a complete lattice. For each $f, g \in L$ and $\alpha < \Omega$, define $f \sqsubseteq_\alpha g$ iff for all $z \in Z$,

- $\forall \beta < \alpha (fz = F_\beta \Leftrightarrow gz = F_\beta \wedge fz = T_\beta \Leftrightarrow gz = T_\beta)$,
- $gz = F_\alpha \Rightarrow fz = F_\alpha \wedge fz = T_\alpha \Rightarrow gz = T_\alpha$.

Then L is a model. The intuition for the definition of the relations \sqsubseteq_α is that $f \sqsubseteq_\alpha g$ iff f and g agree below ‘stratum’ α , and at stratum α and above, f is either below or equivalent to g in the sense that if for some z , $fz = T_\alpha$, then $gz = T_\alpha$ and if $gz = F_\alpha$ then $fz = F_\alpha$.

It is clear that the first two axioms hold. To see that Ax3 holds, suppose that $\alpha < \Omega$, $g \in V^Z$ and $G \subseteq (g)_\alpha$. Then for all $f \in G$, $fz = gz$ whenever gz is in $\{F_\beta, T_\beta : \beta < \alpha\}$. The function $h = \bigsqcup_\alpha G$ is given by $hz = gz$ if $gz \in \{F_\beta, T_\beta : \beta < \alpha\}$. If this condition does not hold, then $hz = T_\alpha$ if there exists $f \in G$ with $fz = T_\alpha$, $hz = F_\alpha$ if $fz = F_\alpha$ for all $f \in G$, and $hz = F_{\alpha+1}$ otherwise.

Finally, Ax4 holds since if $H \subseteq V^Z$ is a nonempty set and $g \in V^Z$ and $\alpha < \Omega$ such that $fz = gz$ for all $f \in H$ and $z \in Z$ when fz or gz is in $\{F_\beta, T_\beta : \beta \leq \alpha\}$, then also $(\bigvee H)z = gz$ whenever gz or $(\bigvee H)z$ is in $\{F_\beta, T_\beta : \beta \leq \alpha\}$.

Example 2.2 Every complete lattice (L, \leq) gives rise to a model. Indeed, define \sqsubseteq_0 to be the relation \leq , and for each $0 < \alpha < \kappa$, define the relation \sqsubseteq_α as the equality relation $=$.

Example 2.3 Suppose that both (L, \leq) and (L, \sqsubseteq_0) are complete lattices. When $0 < \alpha < \kappa$, let \sqsubseteq_α be the equality relation $=$. As before, for each $X \subseteq L$, let $\bigvee X$ denote the supremum of X w.r.t. \leq . Moreover, let $\bigsqcup_0 X$ denote the supremum of X w.r.t. \sqsubseteq_0 . Then L is a model iff one of the following three conditions holds:

1. For all $X \subseteq L$ and $y \in L$, if $X \sqsubseteq_0 y$ then $\bigsqcup_0 X \leq y$.
2. For all $x, y \in L$, if $x \sqsubseteq_0 y$ then $x \leq y$.
3. For all $X \subseteq L$, $\bigvee X \leq \bigsqcup_0 X$.

Indeed, these conditions are equivalent. If the first condition holds, then for all x, y , if $x \sqsubseteq_0 y$ then $x = \bigsqcup_0 \{x\} \leq y$. Suppose that the second condition holds. Then for all X , every upper bound of X w.r.t. \sqsubseteq_0 is an upper bound of X w.r.t. \leq , hence $\bigvee X \leq \bigsqcup_0 X$. And if this holds and $x \sqsubseteq_0 y$, then $\bigvee \{x, y\} \leq \bigsqcup_0 \{x, y\} = y$, hence $x \leq y$. Thus, the second and third conditions are equivalent. Assume finally that the second and third conditions hold and $X \sqsubseteq_0 y$. Then $\bigsqcup_0 X \sqsubseteq_0 y$ and thus $\bigsqcup_0 X \leq y$.

Now, axioms Ax1–Ax4 clearly hold with the possible exception of Ax3 for $\alpha = 0$. But if the above three equivalent conditions are satisfied, then Ax3 holds for $\alpha = 0$, since for any X , $\bigsqcup_0 X$ is just that element whose existence is required in Ax3. Finally, if L is a model, then the first of the three conditions holds by Ax3.

Below we will often denote a model $(L, \leq, (\sqsubseteq_\alpha)_{\alpha < \kappa})$ by just L .

Remark 2.4 It is clear that in any model L , the element z in Ax3 is uniquely determined by x , X and α , and in fact by X and α if X is not empty. We denote it by $\bigsqcup_\alpha X$ and will freely use this notation without specifying x whenever X is not empty. Given X and α , there exists some x with $X \subseteq (x)_\alpha$ iff $y =_\beta y'$ for all $y, y' \in X$ and $\beta < \alpha$, and when X is not empty, $\bigsqcup_\alpha X$ is the unique element z with $X \sqsubseteq_\alpha z$ and such that for all y with $X \sqsubseteq_\alpha y$, both $z \sqsubseteq_\alpha y$ and $z \leq y$. However, when X is empty, then $\bigsqcup_\alpha X$ depends on x , at least when $\alpha > 0$, since it is required to be in $(x)_\alpha$. When $X = \emptyset$, $\bigsqcup_\alpha X$ is the unique element z with $z =_\beta x$ for all $\beta < \alpha$ and such that whenever $y =_\beta x$ for all $\beta < \alpha$, then $z \sqsubseteq_\alpha y$ and $z \leq y$. Hence $\bigsqcup_\alpha \emptyset$ is both the \leq -least element of $(x)_\alpha$ and a \sqsubseteq_α -least element of $(x)_\alpha$.

Note that when L is a model, $x \in L$ and $\alpha < \kappa$, then $\bigsqcup_\alpha \{x\}$ is the \leq -least $y \in L$ with $x \sqsubseteq_\alpha y$ and also the \leq -least $y \in L$ with $x =_\alpha y$, i.e., $\bigsqcup_\alpha \{x\}$ is the \leq -least element of $[x]_\alpha = \{y : y =_\alpha x\}$. Below we will use the notation $x|_\alpha = \bigsqcup_\alpha \{x\}$ for all $x \in L$ and $\alpha < \kappa$. For example, $\perp|_\alpha = \perp$ for all $\alpha < \kappa$, since $\perp =_\alpha \perp$ and $\perp \leq x$ for all $x \in L$. By the above, $x =_\alpha x|_\alpha$ and $x =_\alpha y$ iff $x|_\alpha =_\alpha y|_\alpha$ iff $x|_\alpha = y|_\alpha$ for all $x, y \in X$ and $\alpha < \kappa$. It then follows by Ax2 that for all $x, y \in L$, $x = y$ iff $x|_\alpha = y|_\alpha$ for all $\alpha < \kappa$. Moreover, if $x \in L$ and $\alpha + 1 < \kappa$ (which always holds when $\alpha < \kappa$ and κ is a limit ordinal), then $x|_\alpha$ is the $\sqsubseteq_{\alpha+1}$ -least element of $[x]_\alpha$. Indeed, $x|_\alpha$ is the \leq -least element of $[x]_\alpha$, whereas the \leq -least element of $[x]_{\alpha+1}$ is $\bigsqcup_{\alpha+1} \emptyset$ (with \emptyset considered as a subset of $(x)_{\alpha+1}$). However, $[x]_\alpha = (x)_{\alpha+1}$, so the two least elements are equal. (See also Lemma 3.7 in [18].)

It is known (see Lemma 3.12 in [18]) that the following conditions are equivalent for each $x \in L$ and $\alpha < \kappa$:

- $x = \bigsqcup_\alpha \{x\}$,
- there exists $y \in L$ with $x = \bigsqcup_\alpha \{y\}$,
- there exists a (nonempty) $X \subseteq L$ with $x = \bigsqcup_\alpha X$.

For later use, we prove:

Lemma 2.5 *Suppose that L is a model, $x \in L$ and $\alpha, \beta < \kappa$. Then $(x|_\alpha)|_\beta = x|_{\min\{\alpha, \beta\}}$.*

Proof. Suppose first that $\alpha \leq \beta$. We clearly have $x|_\alpha =_\beta (x|_\alpha)|_\beta$. Moreover, if $x|_\alpha \sqsubseteq_\beta y$ then $x =_\alpha x|_\alpha \sqsubseteq_\alpha y$ by Ax1, hence $x|_\alpha \leq y$. We conclude that $x|_\alpha = (x|_\alpha)|_\beta$. Now let $\beta < \alpha$. Since $x =_\alpha x|_\alpha$ and $\beta < \alpha$, by Ax1 it holds that $x =_\beta x|_\alpha$. Hence for all $y \in L$, we have $x \sqsubseteq_\beta y$ iff $x|_\alpha \sqsubseteq_\beta y$. It follows that $x|_\beta = (x|_\alpha)|_\beta$. \square

Lemma 2.6 *Suppose that L is a model, $x \in L$ and $\alpha < \kappa$, and let $X_i \subseteq (x)_\alpha$ for all $i \in I$. Then $\bigcup_{i \in I} X_i$ and $\{\bigsqcup_\alpha X_i : i \in I\}$ are subsets of $(x)_\alpha$ and thus of $(x)_\beta$ for all $\beta < \alpha$. The following associativity property holds for all $\beta \leq \alpha$:*

$$\bigsqcup_\beta \{\bigsqcup_\alpha X_i : i \in I\} = \bigsqcup_\beta \bigcup_{i \in I} X_i. \quad (1)$$

Proof. When $\beta = \alpha$ this is due to the fact that for all $z \in L$, $\{\bigsqcup_\alpha X_i : i \in I\} \sqsubseteq_\alpha z$ iff $\bigcup_{i \in I} X_i \sqsubseteq_\alpha z$. Note that when I is empty, then both sides of (1) are equal to the \leq -least element of $(x)_\alpha$.

Suppose now that $\beta < \alpha$. Then $x =_\beta y$ holds for all $y \in \bigcup_{i \in I} X_i$, moreover, $x =_\beta \bigsqcup_\alpha X_i$ for all $i \in I$. Hence, when I is not empty, both sides of (1) are equal to the \leq -least element of $(x)_\beta$. Otherwise, if I is empty, both sides are equal to the \leq -least element of $(x)_\beta$. \square

In [18], it is proved that for any sequence $(x_\alpha)_{\alpha < \kappa}$ in a model L , there exists some $x \in L$ with $x_\alpha = x|_\alpha$ for all $\alpha < \kappa$ iff

- $x_\alpha =_\alpha x_\beta$ for all $\alpha < \beta < \kappa$, and
- x_α is the \leq -least element of $[x_\alpha]_\alpha$, for all $\alpha < \kappa$.

It follows that $x_\alpha \leq x_\beta$ for all $\alpha \leq \beta < \kappa$, and if $\alpha + 1 < \kappa$, then x_α is a $\sqsubseteq_{\alpha+1}$ -least element of $[x_\alpha]_\alpha$. Such sequences are called *compatible*. The element x is uniquely determined by the compatible sequence $(x_\alpha)_{\alpha < \kappa}$. It is given by $x = \bigvee_{\alpha < \kappa} x_\alpha$. Indeed, if $\alpha < \kappa$, then $x = \bigvee_{\alpha \leq \gamma < \kappa} x_\gamma =_\alpha x_\alpha$ by Ax4, since $x_\alpha =_\alpha x_\gamma$ for all $\alpha \leq \gamma < \kappa$. Hence $x|_\alpha = x_\alpha|_\alpha = x_\alpha$ for all $\alpha < \kappa$.

For later use we prove:

Lemma 2.7 *Suppose that the sequence $(x_\gamma)_{\gamma < \kappa}$ is compatible and $x = \bigvee_{\gamma < \kappa} x_\gamma$, so that $x =_\gamma x_\gamma$ for all $\gamma < \kappa$. Then for each limit ordinal $\alpha < \kappa$, $y = \bigvee_{\gamma < \alpha} x_\gamma$ is both the \leq -least and a \sqsubseteq_α -least element of the set $(x)_\alpha = (x_\alpha)_\alpha$.*

Proof. Since the sequence $(x_\gamma)_{\gamma < \kappa}$ is compatible, it is increasing w.r.t. \leq . Let $\alpha < \kappa$ be a limit ordinal. Then for each $\beta < \alpha$, $y = \bigvee_{\gamma < \alpha} x_\gamma = \bigvee_{\beta \leq \gamma < \alpha} x_\gamma$. By compatibility, also $x_\beta = x_\gamma$ whenever $\beta \leq \gamma < \alpha$, hence by Ax4, $y =_\beta x_\beta$ for all $\beta < \alpha$, proving $y \in (x)_\alpha$. If $z \in (x)_\alpha$, then $z \in [x_\gamma]_\gamma = [x]_\gamma$ for all $\gamma < \alpha$. But each x_γ is \leq -least in $[x_\gamma]_\gamma$, hence $x_\gamma \leq z$. Since this holds for all γ , $y = \bigvee_{\gamma < \alpha} x_\gamma \leq z$. Thus, y is the \leq -least element of $(x)_\alpha$, hence also a \sqsubseteq_α -least element of $(x)_\alpha$ (cf. Axiom 3 in the case when X is empty). \square

For all $x, y \in L$ and $\alpha < \kappa$, let us write $x \sqsubset_\alpha y$ to denote that $x \sqsubseteq_\alpha y$ and $x \neq y$. Moreover, we define $x \sqsubset y$ iff there is some α with $x \sqsubset_\alpha y$, and let $x \sqsubseteq y$ iff $x \sqsubset y$ or $x = y$.

The following result was proved in [18].

Theorem 2.8 *Suppose that L is a model. Then (L, \sqsubseteq) is a complete lattice.*

Remark 2.9 *Suppose that L is a model. As usual, let \perp denote the least element of L w.r.t. \leq . Then \perp is also the least element of L w.r.t. \sqsubseteq . Indeed, for each $\alpha < \kappa$, $\perp \in (\perp)_\alpha$ and if $y \in (\perp)_\alpha$ then $\perp \leq y$ (as this inequality holds for all y). Hence, $\bigsqcup_\alpha \{\emptyset\} = \perp$ with \emptyset considered as a subset of $(\perp)_\alpha$. Suppose now that $x \in L$, $x \neq \perp$. Then there is a least ordinal $\alpha < \kappa$ with $\perp \neq_\alpha x$. Then $x \in (\perp)_\alpha$, hence $\perp \sqsubseteq_\alpha x$, and since $\perp \neq_\alpha x$, we have $\perp \sqsubset_\alpha x$. Thus, for each $x \in L$, either $x = \perp$ or $\perp \sqsubset_\alpha x$ for some $\alpha < \kappa$. We conclude that $\perp \sqsubset x$. (It follows that $\perp \sqsubseteq_0 x$ for all x .)*

Example 2.10 *In the standard model V^Z defined in Example 2.1, the greatest element w.r.t. \sqsubseteq is the function mapping each $z \in Z$ to T_0 , which is also the greatest element with respect to \leq . However, the greatest elements with respect to the orderings need not be the same, not even in finite models. Consider the lattice $L = \{0, 1\}^2$ ordered by the relation \leq as usual. Let $\kappa = 2$ and define $x_1 x_2 \sqsubseteq_0 y_1 y_2$ iff $x_1 = y_1$ or $x_1 \leq y_1$. Moreover, let $x_1 x_2 \sqsubseteq_1 y_1 y_2$ iff $x_1 = y_1 = 0$ and $x_2 \leq y_2$ or $x_1 = y_1 = 1$ and $x_2 \geq y_2$. Then L is a model and the greatest elements of L w.r.t. \leq and \sqsubseteq are 11 and 10, respectively.*

We now define α -monotonic and α -continuous functions, where $\alpha < \kappa$. Suppose that L and L' are models. We say that a function $f : L \rightarrow L'$ is α -monotonic if it preserves the preordering \sqsubseteq_α , i.e., when $x \sqsubseteq_\alpha y$ implies $fx \sqsubseteq_\alpha fy$ for all $x, y \in L$. Moreover, we say that f is α -continuous if it is α -monotonic and for all nonempty linearly ordered sets (I, \leq) and $x_i \in L$ for $i \in I$, if $x_i \sqsubseteq_\alpha x_j$ for all $i \leq j$ in I , then

$$f\left(\bigsqcup_\alpha \{x_i : i \in I\}\right) =_\alpha \bigsqcup_\alpha \{fx_i : i \in I\},$$

or equivalently,

$$(f(\bigsqcup_\alpha \{x_i : i \in I\}))|_\alpha = \bigsqcup_\alpha \{fx_i : i \in I\}.$$

(Note that since $x_i =_\beta x_j$ for all $\beta < \alpha$ and $i, j \in I$ and f is α -monotonic, also $fx_i =_\beta fx_j$ for all $\beta < \alpha$ and $i, j \in I$, hence $\bigsqcup_\alpha \{x_i : i \in I\}$ and $\bigsqcup_\alpha \{fx_i : i \in I\}$ exist.)

Example 2.11 *Suppose that L is a complete lattice viewed as a model as in Example 2.2. Let $\alpha < \kappa$. Then a function $f : L \rightarrow L$ is 0-monotonic iff it is α -monotonic for all $\alpha < \kappa$ iff f is monotonic with respect to \leq , and f is 0-continuous iff α -continuous for all $\alpha < \kappa$ iff f is continuous with respect to \leq , or simply just continuous: $f(\bigvee X) = \bigvee f(X)$ for all nonempty linearly ordered sets $X \subseteq L$ w.r.t. \leq .*

Remark 2.12 *A function between models that is α -monotonic or α -continuous for all $\alpha < \kappa$ is not necessarily monotonic w.r.t. the relation \sqsubseteq . See [18].*

We will make use of the next lemma without explicitly mentioning it.

Lemma 2.13 *Suppose that L, L', L'' are models and $f : L \rightarrow L'$ and $g : L' \rightarrow L''$ are α -monotonic (resp. α -continuous), where α is an ordinal less than κ . Then the function $g \circ f : L \rightarrow L''$ is also α -monotonic (resp. α -continuous). Moreover, the identity function $\text{id}_L : L \rightarrow L$ is α -continuous as is any constant function $L \rightarrow L'$.*

Let \mathbf{Mod}_m (resp. \mathbf{Mod}_c) denote the category of models and those functions between them which are α -monotonic (resp. α -continuous) for all $\alpha < \kappa$.

Theorem 2.14 *The categories \mathbf{Mod}_m and \mathbf{Mod}_c are cc's.*

Proof. We need to show that \mathbf{Mod}_m (resp. \mathbf{Mod}_c) has a terminal object and binary products. In fact, it is easy to see that \mathbf{Mod}_m (resp. \mathbf{Mod}_c) has all products and that products can be constructed pointwise. Thus, if $L_i = (L_i, \leq_i, (\sqsubseteq_{i,\alpha})_{\alpha < \kappa})$ is a model for each $i \in I$, where I is any set, then the cartesian product $L = \prod_{i \in I} L_i$, equipped with the pointwise order relations \leq and \sqsubseteq_α , $\alpha < \kappa$, defined for $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ in L by $x \leq y$ iff $x_i \leq_i y_i$ for all $i \in I$ and $x \sqsubseteq_\alpha y$ iff $x_i \sqsubseteq_{i,\alpha} y_i$ for all $i \in I$, is also a model. It follows that $\bigvee X$ can be computed pointwise for all $X \subseteq L$. A similar fact is true for $\bigsqcup_\alpha X$ whenever $X \subseteq (x)_\alpha$ for some $x \in L$ and $\alpha < \kappa$.

In both categories, the projections $\pi_{L_j}^{\prod_{i \in I} L_i} : \prod_{i \in I} L_i \rightarrow L_j$, for $j \in I$, are the usual projection functions. \square

Remark 2.15 *Suppose that L_i is a model as above for each $i \in I$. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ in the product model $L = \prod_{i \in I} L_i$. Then $x \sqsubseteq y$ in L iff either $x = y$, or there is some $\alpha < \kappa$ such that for all $i \in I$, $x_i \sqsubseteq_{i,\alpha} y_i$, moreover, there is some $j \in I$ with $x_j \sqsubset_{j,\alpha} y_j$.*

Below we will often make use of the following simple fact.

Lemma 2.16 *Suppose that L, L', L'' are models and $\alpha < \kappa$. Then a function $f : L' \times L'' \rightarrow L$ is α -monotonic (resp. α -continuous) iff the following conditions hold:*

- For each fixed $x \in L'$, the function $x_f : L'' \rightarrow L$ defined by $x_f y = f(x, y)$ is α -monotonic (α -continuous).
- For each fixed $y \in L''$, the function $f_y : L' \rightarrow L$ defined by $f_y x = f(x, y)$ is α -monotonic (α -continuous).

Example 2.17 [18, 24] *Consider the linearly ordered complete lattice V of truth values of Example 2.1 and let Z be a set. Define \vee and \wedge as the binary supremum and infimum operations on V , and define $\neg : V \rightarrow V$ by $\neg F_\alpha = T_{\alpha+1}$, $\neg T_\alpha = F_{\alpha+1}$, for all $\alpha < \Omega$, and $\neg 0 = 0$. Extend these operations to V^Z pointwise, so that $(f \vee g)z = fz \vee gz$ for all $z \in Z$, etc. Then \vee, \wedge, \neg are α -continuous functions over V^Z for all $\alpha < \Omega$.*

3 Stratified least fixed points

In this section we recall a fixed point theorem (Theorem 3.1) from [18] involving those functions over a model L which are α -monotonic for all $\alpha < \kappa$. (Recall that κ is a fixed nonzero ordinal.) Then we extend this operation to a *parametrized fixed point operation*

$$f : L \times L' \mapsto f^\dagger : L' \rightarrow L,$$

where L and L' are models and f is α -monotonic for all $\alpha < \kappa$, and prove that f^\dagger is also α -monotonic for all $\alpha < \kappa$. Moreover, we prove that when f is α -continuous for all $\alpha < \kappa$, then so is f^\dagger .

Theorem 3.1 *Suppose that L is a model and $f : L \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. Then f has a least pre-fixed point with respect to the partial order \sqsubseteq which is also a fixed point.*

Thus, the theorem asserts that there is some $x \in L$ with $fx \sqsubseteq x$ and such that for all $y \in L$, if $fy \sqsubseteq y$ then $x \sqsubseteq y$. Moreover, x is a fixed point, i.e., $fx = x$. In particular, x is the unique least fixed point of f w.r.t. the order relation \sqsubseteq . The proof of Theorem 3.1 in [18] provides a construction of the least fixed point by a two level transfinite sequence of approximations. We will describe the construction in more

detail below. Since every α -continuous function is α -monotonic, the theorem also applies to functions that are α -continuous for all $\alpha < \kappa$. For such functions, the inner level of the transfinite sequence of approximations terminates in ω steps, where ω denotes the first infinite ordinal.

We will be concerned with parametrized fixed points. Suppose that L, L' are models and $f : L \times L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. Then for each fixed $y \in L'$, the function $f_y : L \rightarrow L$ defined by $f_y x = f(x, y)$ is also α -monotonic for all $\alpha < \kappa$ and thus by Theorem 3.1 has a least (pre-)fixed point with respect to the ordering \sqsubseteq . Let us denote this least (pre-)fixed point by $f^\dagger y$. Then f^\dagger , as a function of x , maps L' into L .

Example 3.2 *As explained in Example 2.2, each complete lattice $L = (L, \leq)$ gives rise to a model for any nonzero ordinal κ . Moreover, by Example 2.11, a function $f : L \rightarrow L'$ between complete lattices L and L' is α -monotonic for all $\alpha < \kappa$ iff it is monotonic with respect to the ordering \leq . Similarly, f is α -continuous for all $\alpha < \kappa$ iff it is continuous w.r.t. \leq . Thus, in this case Theorem 3.1 asserts that for a complete lattice (L, \leq) , every monotonic function $L \rightarrow L$ has a least pre-fixed point (w.r.t. \leq) which is a fixed point. This is a part of the Knaster-Tarski fixed point theorem, see [7, 27]. In particular, for all complete lattices L, L' and monotonic functions $f : L \times L' \rightarrow L$ and $y \in L'$, $f^\dagger y$ is the least (pre-)fixed point of the function $f_y : L \rightarrow L$ mapping each $x \in L$ to $f(x, y)$.*

Example 3.3 [18, 24] *Suppose that Z is a denumerable set of propositional variables and P is an at most countably infinite propositional logic program over Z , possibly involving negation. Thus P is a countable set of instructions of the form $z \leftarrow \ell_1 \wedge \dots \wedge \ell_k$, where $z \in Z$ and ℓ_i is a literal for each i . Consider the model $L = V^Z$ of ‘interpretations’ defined earlier, cf. Example 2.1. Then P induces a function $f_P : L \rightarrow L$ which maps an interpretation $I \in L$ to the interpretation $J = f_P(I)$ such that $J(z) = \bigvee_{z \leftarrow \ell_1 \wedge \dots \wedge \ell_k \in P} (I(\ell_1) \wedge \dots \wedge I(\ell_k))$, where for a negative literal $\ell = \neg y$ and $\alpha < \Omega$, $I(\ell) = T_{\alpha+1}$ if $I(y) = F_\alpha$, $I(\ell) = F_{\alpha+1}$ if $I(y) = T_\alpha$, and $I(\ell) = 0$ if $I(y) = 0$. Then f_P is α -monotonic for all $\alpha < \Omega$. The semantics of P is defined in [24] as the least fixed point of f_P w.r.t. \sqsubseteq .*

For example, consider the program P :

$$\begin{aligned} p &\leftarrow \neg q \\ q &\leftarrow \neg r \\ s &\leftarrow p \\ s &\leftarrow \neg s \\ t &\leftarrow \end{aligned}$$

Then the least fixed point of f_P w.r.t. \sqsubseteq is: $(r, F_0), (q, T_1), (p, F_2), (s, 0), (t, T_0)$. Intuitively q is ‘less true’ than t , since q is true only because r is false by default, while there is an instruction declaring t to be true. This is reflected by the least fixed point.

The construction of $f^\dagger y$ mentioned above makes use of the following lemma from [18], slightly adjusted to the parametrized setting.

Lemma 3.4 *Suppose that L, L' are models and $f : L \times L' \rightarrow L$ is α -monotonic, where $\alpha < \kappa$. If $x \in L$, $y \in L'$ and $\alpha < \kappa$ with $x \sqsubseteq_\alpha f(x, y)$, then there is some $z \in L$ with the following properties:*

- $x \sqsubseteq_\alpha z =_\alpha f(z, y)$,
- if $z' \in L$ with $x \sqsubseteq_\alpha z'$ and $f(z', y) \sqsubseteq_\alpha z'$, then $z \sqsubseteq_\alpha z'$,
- z is the \leq -least element of the set $[z]_\alpha$, and if $\alpha + 1 < \kappa$, then z is also a $\sqsubseteq_{\alpha+1}$ -least element of $[z]_\alpha$.

It follows that z is uniquely determined as a function of x and y and we denote it by $f_\alpha(x, y)$. Indeed, if z and z' both satisfy the above conditions, then $z \sqsubseteq_\alpha z'$ and $z' \sqsubseteq_\alpha z$, hence $z =_\alpha z'$ and $[z]_\alpha = [z']_\alpha$, so that z and z' are the \leq -least elements of the same set. Moreover, if $\alpha + 1 < \kappa$, then $z \sqsubseteq_{\alpha+1} f(z, y)$, since z is a $\sqsubseteq_{\alpha+1}$ -least element of $[z]_\alpha = [f(z, y)]_\alpha$.

The element $z = f_\alpha(x, y)$ can be constructed by approximating it with the following sequence $(x_\gamma)_\gamma$, where γ ranges over the ordinals. Let $x_0 = x$ and $x_\gamma = f(x_\delta, y)$ when $\gamma = \delta + 1$ is a successor ordinal. When γ is a limit ordinal, define $x_\gamma = \bigsqcup_\alpha \{x_\delta : \delta < \gamma\}$. Then $x_\beta \sqsubseteq_\alpha x_\gamma$ for all ordinals β and γ with $\beta < \gamma$. Thus there is a least ordinal λ_0 with $x_{\lambda_0} =_\alpha x_{\lambda_0+1}$. It follows that $x_\beta =_\alpha x_\gamma$ for all β and γ with $\lambda_0 \leq \beta < \gamma$. The element z is x_λ for the least limit ordinal λ with $\lambda_0 \leq \lambda$. In the case when f is γ -continuous for all $\gamma < \kappa$, the ordinal λ is ω , so that the construction stops in ω steps.

Now $z = f^\dagger y$ can be constructed as follows. For each $\alpha < \kappa$, let $x_\alpha = \bigvee_{\beta < \alpha} z_\beta$ and $z_\alpha = f_\alpha(x_\alpha, y)$, so that $x_0 = \perp$. This construction is legitimate, since as shown in [18], $x_\alpha \sqsubseteq_\alpha f(x_\alpha, y)$ for all $\alpha < \kappa$. Moreover, the sequence $(z_\alpha)_{\alpha < \kappa}$ is compatible and $z = \bigvee_{\alpha < \kappa} z_\alpha$, so that $z|_\alpha = z_\alpha$ for all $\alpha < \kappa$.

Remark 3.5 Sometimes we will apply the dagger operation to functions $f : L \rightarrow L$, where L is a model and f is α -monotonic for all $\alpha < \kappa$. In this case we identify L with $L \times \mathbf{1}$, where $\mathbf{1}$ is a fixed one-element model, so that $f^\dagger : \mathbf{1} \rightarrow L$, which is in turn conveniently identified with an element of L .

We will make use of the following lemmas concerning the functions f_α .

Lemma 3.6 Suppose that L, L' are models and $f : L \times L' \rightarrow L$ is α -monotonic, where $\alpha < \kappa$. Suppose that $x, x' \in L$ and $y, y' \in L'$ with $x \sqsubseteq_\alpha x'$ and $y \sqsubseteq_\alpha y'$, moreover, $x \sqsubseteq_\alpha f(x, y)$ and $x' \sqsubseteq_\alpha f(x', y')$. Let $z = f_\alpha(x, y)$ and $z' = f_\alpha(x', y')$. Then $z \sqsubseteq_\alpha z'$. And if $x =_\alpha x'$ and $y =_\alpha y'$ then $z =_\alpha z'$, and in fact $z = z'$.

Proof. First note that since $x \sqsubseteq_\alpha f(x, y)$ and $x' \sqsubseteq_\alpha f(x', y')$, both $z = f_\alpha(x, y)$ and $z' = f_\alpha(x', y')$ exist. Since f is α -monotonic, we have $f(z', y) \sqsubseteq_\alpha f(z', y') \sqsubseteq_\alpha z'$. Also, $x \sqsubseteq_\alpha x' \sqsubseteq_\alpha z'$. It follows by the 2nd clause of Lemma 3.4 that $z \sqsubseteq_\alpha z'$. Suppose now that $x =_\alpha x'$ and $y =_\alpha y'$. Then $z \sqsubseteq_\alpha z'$ and $z' \sqsubseteq_\alpha z$, thus $z =_\alpha z'$. Since z is the \leq -least element of $[z]_\alpha$ and z' is the \leq -least element of $[z']_\alpha$, and since $[z]_\alpha = [z']_\alpha$, it follows that $z = z'$. \square

Lemma 3.7 Suppose that L, L' are models and $f : L \times L' \rightarrow L$ is α -continuous, where $\alpha < \kappa$. Suppose that (I, \leq) is a nonempty linearly ordered set and $x_i \in L, y_i \in L'$ for all $i \in I$ such that $x_i \sqsubseteq_\alpha x_j$ and $y_i \sqsubseteq_\alpha y_j$ whenever $i \leq j$ in I , moreover, $x_i \sqsubseteq_\alpha f(x_i, y_i)$ for all $i \in I$. Then

$$\bigsqcup_\alpha \{x_i : i \in I\} \sqsubseteq_\alpha f(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\})$$

and

$$f_\alpha(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\}) = \bigsqcup_\alpha \{f_\alpha(x_i, y_i) : i \in I\}.$$

Proof. First, since $x_i \sqsubseteq_\alpha f(x_i, y_i)$ for all $i \in I$ and f is α -continuous,

$$\begin{aligned} \bigsqcup_\alpha \{x_i : i \in I\} &\sqsubseteq_\alpha \bigsqcup_\alpha \{f(x_i, y_i) : i \in I\} \\ &=_\alpha f(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\}). \end{aligned}$$

Thus, $f_\alpha(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\})$ exists.

Define $f^n : L \times L' \rightarrow L$ for $n \geq 0$ by $f^0 = \pi_{L \times L'}^{L \times L'}$ and $f^n = f \circ \langle f^{n-1}, \pi_{L' \times L'}^{L \times L'} \rangle$ for $n > 0$. Thus, for all $x \in L$ and $y \in L'$, $f^0(x, y) = x$ and $f^n(x, y) = f(f^{n-1}(x, y), y)$, for all $n > 0$. Since f is α -continuous, so is f^n for all $n \geq 0$. It follows from our assumptions that $f^n(x_i, y_i) =_\beta f^m(x_j, y_j)$ for all $n, m \geq 0$, $i, j \in I$ and $\beta < \alpha$. Moreover, using α -continuity,

$$\begin{aligned} f_\alpha(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\}) &= \bigsqcup_\alpha \{f^n(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\}) : n \geq 0\} \\ &=_\alpha \bigsqcup_\alpha \{f^n(x_i, y_i) : i \in I, n \geq 0\}. \end{aligned}$$

However, by Lemma 3.4, $f_\alpha(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\})$ is the \leq -least element of $[f_\alpha(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\})]_\alpha$ and by definition, $\bigsqcup_\alpha \{f^n(x_i, y_i) : i \in I, n \geq 0\}$ is the \leq -least element of $[\bigsqcup_\alpha \{f^n(x_i, y_i) : i \in I, n \geq 0\}]_\alpha$, hence

$$f_\alpha(\bigsqcup_\alpha \{x_i : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\}) = \bigsqcup_\alpha \{f^n(x_i, y_i) : i \in I, n \geq 0\}.$$

Also,

$$\begin{aligned} \bigsqcup_\alpha \{f_\alpha(x_i, y_i) : i \in I\} &= \bigsqcup_\alpha \{\bigsqcup_\alpha \{f^n(x_i, y_i) : n \geq 0\} : i \in I\} \\ &= \bigsqcup_\alpha \{f^n(x_i, y_i) : i \in I, n \geq 0\}, \end{aligned}$$

by Lemma 2.6. □

Now we can prove that if f is α -monotonic or α -continuous for all $\alpha < \kappa$, then so is f^\dagger .

Proposition 3.8 *Suppose that L, L' are models and $f : L \times L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. Then the function $f^\dagger : L' \rightarrow L$ is also α -monotonic for all $\alpha < \kappa$. And if f is α -continuous for all $\alpha < \kappa$, then the same holds for f^\dagger .*

Proof. Suppose first that f is α -monotonic for all $\alpha < \kappa$. Let $y \sqsubseteq_\alpha y'$ in L' , where $\alpha < \kappa$, and denote $z = f^\dagger y$ and $z' = f^\dagger y'$. We want to show that $z \sqsubseteq_\alpha z'$.

For each $\gamma < \kappa$, let $x_\gamma = \bigvee_{\delta < \gamma} z_\delta$ and $z_\gamma = f_\gamma(x_\gamma, y)$. Symmetrically, let $x'_\gamma = \bigvee_{\delta < \gamma} z'_\delta$ and $z'_\gamma = f_\gamma(x'_\gamma, y')$. Thus $x_0 = x'_0 = \perp$. We know that $x_\gamma \sqsubseteq_\gamma f(x_\gamma, y)$ and $x'_\gamma \sqsubseteq_\gamma f(x'_\gamma, y')$ for all $\gamma < \kappa$. Moreover, the sequences $(z_\gamma)_{\gamma < \kappa}$ and $(z'_\gamma)_{\gamma < \kappa}$ are compatible and $z_\gamma = z|_\gamma$ and $z'_\gamma = z'|_\gamma$ for all $\gamma < \kappa$, and $z = \bigvee_{\gamma < \kappa} z_\gamma$ and $z' = \bigvee_{\gamma < \kappa} z'_\gamma$. Thus, $z =_\alpha z_\alpha$ and $z' =_\alpha z'_\alpha$, so that $z \sqsubseteq_\alpha z'$ holds exactly when $z_\alpha \sqsubseteq_\alpha z'_\alpha$.

It follows by induction on β using Lemma 3.6 that $z_\beta = z'_\beta$ and $x_\beta = x'_\beta$ for all $\beta < \alpha$. Indeed, $x_0 = x'_0 = \perp$ and if $\alpha > 0$ then $z_0 = z'_0$ by Lemma 3.6 since $y =_0 y'$. And if $0 < \beta < \alpha$ and the claim holds for all $\gamma < \beta$, then $x_\beta = \bigvee_{\gamma < \beta} z_\gamma = \bigvee_{\gamma < \beta} z'_\gamma = x'_\beta$, and then $z_\beta = z'_\beta$ by Lemma 3.6 since $y =_\beta y'$. Also, $x_\alpha = \bigvee_{\beta < \alpha} z_\beta = \bigvee_{\beta < \alpha} z'_\beta = x'_\alpha$. Since $y \sqsubseteq_\alpha y'$, it follows now that $z_\alpha = f_\alpha(x_\alpha, y) \sqsubseteq_\alpha f_\alpha(x'_\alpha, y') = z'_\alpha$.

Suppose next that f is α -continuous for all $\alpha < \kappa$. We prove that f^\dagger is also α -continuous for all $\alpha < \kappa$. To this end, let $\alpha < \kappa$, (I, \leq) be a nonempty linearly ordered set and $y_i \in L$ for all $i \in I$ such that $y_i \sqsubseteq_\alpha y_j$ whenever $i, j \in I$ with $i \leq j$. Let $y = \bigsqcup_\alpha \{y_i : i \in I\}$. For each $i \in I$ and $\gamma < \kappa$, define $x_{i,\gamma} = \bigvee_{\beta < \gamma} z_{i,\beta}$ and $z_{i,\gamma} = f_\gamma(x_{i,\gamma}, y_i)$. Moreover, define $x_\gamma = \bigvee_{\beta < \gamma} z_\beta$ and $z_\gamma = f_\gamma(x_\gamma, y)$. We know that $x_{i,\gamma} \sqsubseteq_\gamma f(x_{i,\gamma}, y_i)$ for all $i \in I$ and $\gamma < \kappa$. Similarly, $x_\gamma \sqsubseteq_\gamma f(x_\gamma, y)$ for all $\gamma < \kappa$.

Let $z = f^\dagger y$ and $z_i = f^\dagger y_i$ for all $i \in I$. We already know that $z = \bigvee_{\gamma < \kappa} z_\gamma$ and $z_i = \bigvee_{\gamma < \kappa} z_{i,\gamma}$ for all $i \in I$. Also, $z_i \sqsubseteq_\alpha z_j \sqsubseteq_\alpha z$ for all $i, j \in I$ with $i \leq j$. We want to prove that $z =_\alpha \bigsqcup_\alpha \{z_i : i \in I\}$. Since $z =_\alpha z_\alpha$ and $z_i =_\alpha z_{i,\alpha}$ for all $i \in I$, this holds if $z_\alpha =_\alpha \bigsqcup_\alpha \{z_{i,\alpha} : i \in I\}$.

It follows by induction using Lemma 3.6 that $x_\gamma = x_{i,\gamma}$ and $z_i = z_{i,\gamma}$ for all $i \in I$ and $\gamma < \alpha$. Indeed, suppose that $\gamma < \alpha$, $x_\delta = x_{i,\delta}$ and $z_\delta = z_{i,\delta}$ for all $i \in I$ and $\delta < \gamma$. Then $x_\gamma = \bigvee_{\delta < \gamma} x_\delta = \bigvee_{\delta < \gamma} x_{i,\delta} = x_{i,\gamma}$ for all $i \in I$. Moreover, since $y =_\gamma y_i$ for all $i \in I$, by Lemma 3.6 also $z_\gamma = f_\gamma(x_\gamma, y) = f_\gamma(x_{i,\gamma}, y_i) = z_{i,\gamma}$ for all $i \in I$. Similarly, $x_\alpha = x_{i,\alpha}$ for all $i \in I$. It follows eventually from Lemma 3.7 that

$$\begin{aligned} z_\alpha &= f_\alpha(x_\alpha, y) \\ &= f_\alpha(\bigsqcup_\alpha \{x_{i,\alpha} : i \in I\}, \bigsqcup_\alpha \{y_i : i \in I\}) \\ &= \bigsqcup_\alpha \{f_\alpha(x_{i,\alpha}, y_i) : i \in I\} \\ &= \bigsqcup_\alpha \{z_{i,\alpha} : i \in I\}. \end{aligned}$$

□

4 The cartesian fixed point identities

An *external dagger operation* [5] on a cartesian category assigns a morphism $f^\dagger : L' \rightarrow L$ to each morphism $f : L \times L' \rightarrow L$. In particular, \mathbf{Mod}_m and \mathbf{Mod}_c are equipped with an external dagger operation. In this section, we prove that with respect to the cartesian structure, the dagger operation on these categories satisfies the standard identities of fixed point operations described by the axioms of iteration theories [4].

We recall that a cartesian category is a category with finite products. We will assume that in each cartesian category, a terminal object $\mathbf{1}$ is fixed, and for each pair of objects L, L' , we assume a fixed product object $L \times L'$ and specified projection morphisms $\pi_L^{L \times L'} : L \times L' \rightarrow L$ and $\pi_{L'}^{L \times L'} : L \times L' \rightarrow L'$. Moreover, we assume that product is ‘associative on the nose’, so that in particular $L \times (L' \times L'') = (L \times L') \times L''$ and

$$\pi_{L''}^{L' \times L''} \circ \pi_{L' \times L''}^{L \times (L' \times L'')} = \pi_{L''}^{(L \times L') \times L''},$$

for all objects L, L', L'' , etc. We identify an object $L \times \mathbf{1}$ with L and a projection $\pi_L^{L \times \mathbf{1}}$ with \mathbf{id}_L .

Some notation. In any cartesian category, for any morphisms $f : L'' \rightarrow L$ and $g : L'' \rightarrow L'$ we denote by $\langle f, g \rangle$ the *pairing* of f and g , ie., the unique morphism $h : L'' \rightarrow L \times L'$ with $f = \pi_L^{L \times L'} \circ h$ and $g = \pi_{L'}^{L \times L'} \circ h$. Note that in \mathbf{Mod}_m or \mathbf{Mod}_c , $\langle f, g \rangle x = (fx, gx)$ for all $x \in L''$. Moreover, for $f : L' \rightarrow L$ and $g : K' \rightarrow K$, we let $f \times g$ denote the morphism $\langle f \circ \pi_{L'}^{L' \times K'}, g \circ \pi_{K'}^{L' \times K'} \rangle : L' \times K' \rightarrow L \times K$. Thus, in \mathbf{Mod}_m or \mathbf{Mod}_c , $(f \times g)(x, y) = (fx, gy)$ for all $(x, y) \in L' \times K'$. These operations are associative. We define the *tupling* $\langle f_1, \dots, f_n \rangle : L' \rightarrow \prod_{i=1}^n L_i$ of morphisms $f_i : L' \rightarrow L_i$, $i = 1, \dots, n$ by repeated applications of the pairing operation.

We now review one of the axiomatizations of iteration categories (or iteration theories) from [4, 11]. (Actually only cartesian categories generated by a single object were treated in [4], but the generalization is straightforward, see eg. [5, 14, 15].)

Fixed point identity

$$f^\dagger = f \circ \langle f^\dagger, \mathbf{id}_{L'} \rangle, \quad f : L \times L' \rightarrow L$$

Parameter identity

$$f^\dagger \circ g = (f \circ (\mathbf{id}_L \times g))^\dagger, \quad f : L \times L' \rightarrow L, \quad g : L'' \rightarrow L'$$

Composition identity

$$(g \circ \langle f, \pi_{L''}^{L \times L''} \rangle)^\dagger = g \circ (\langle f \circ \langle g, \pi_{L''}^{L' \times L''} \rangle^\dagger, \mathbf{id}_{L''} \rangle), \quad (2)$$

where $f : L \times L'' \rightarrow L'$ and $g : L' \times L'' \rightarrow L$.

Double dagger identity

$$(f \circ (\Delta_L \times \mathbf{id}_{L'}))^\dagger = f^{\dagger\dagger}, \quad f : L \times L \times L' \rightarrow L$$

(Here, Δ_L is the diagonal morphism $\langle \mathbf{id}_L, \mathbf{id}_L \rangle : L \rightarrow L \times L$.)

Commutative identities

$$\pi \circ \langle f \circ (\rho_1 \times \mathbf{id}_{L'}), \dots, f \circ (\rho_n \times \mathbf{id}_{L'}) \rangle^\dagger = (f \circ (\Delta_L^n \times \mathbf{id}_{L'}))^\dagger$$

where $n > 1$, $f : L^n \times L' \rightarrow L$, $\Delta_L^n = \langle \mathbf{id}_L, \dots, \mathbf{id}_L \rangle$ is the diagonal morphism $L \rightarrow L^n$, the $\rho_i : A^n \rightarrow A$ are tuplings of projections, and π denotes the first projection $L^n \rightarrow L$.¹

Following [4], we say that a cartesian category equipped with an external dagger operation is a *Conway category* (or *Conway theory*) if it satisfies the parameter, composition and double dagger identities.

¹Actually the above identities are an equivalent form of the original commutative identities [11].

Moreover, we say that a cartesian category equipped with an external dagger operation is an *iteration category* (or an *iteration theory*) if it is a Conway category satisfying the commutative identities.

Before proceeding to prove that \mathbf{Mod}_m and \mathbf{Mod}_c are iteration categories, we recall some facts from [4]. It is clear that the fixed point identity is an instance of the composition identity. Also, Conway categories satisfy several other well-known identities including the pairing identity (or Bekić identity) [2, 25]. It is immediately clear that in Conway categories, the commutative identities are implied by the following quasi-identity.

Weak functorial dagger

$$f \circ (\Delta_L^n \times \mathbf{id}_{L'}) = \Delta_L^n \circ g \Rightarrow f^\dagger = \Delta_L^n \circ g^\dagger,$$

where $f : L^n \times L' \rightarrow L^n$, $g : L \times L' \rightarrow L$ and Δ_L^n denotes the diagonal morphism $L \rightarrow L^n$. For simplifications of the commutative identities, we refer to [13, 14, 15].

Remark 4.1 Consider the category \mathcal{C} of complete lattices and monotonic or continuous functions. By Example 3.2, the least fixed point operation is an external dagger operation on \mathcal{C} . It is known that each of the above identities as well as the weak functorial dagger implication holds in \mathcal{C} . Moreover, as shown in [4, 11], an identity involving the cartesian operations and dagger holds in \mathcal{C} iff it holds in all iteration categories. For a generalization of this completeness result involving partially ordered sets and monotonic functions with enough least fixed points or least pre-fixed points, see [12]. For initial fixed points we refer to [5, 17].

Theorem 4.2 \mathbf{Mod}_m is an iteration category with a weak functorial dagger.

Proof. It is clear from the definition of dagger that the fixed point identity holds. Due to the ‘pointwise’ definition of dagger, the parameter identity also holds. Indeed, let $f : L \times L' \rightarrow L$ and $g : L'' \rightarrow L'$, where L, L', L'' are models and f, g are α -monotonic for all $\alpha < \kappa$. We want to show that $(f \circ (\mathbf{id}_L \times g))^\dagger = f^\dagger \circ g$. To this end, let $z \in L''$. By definition, $(f \circ (\mathbf{id}_L \times g))^\dagger z$ is the \sqsubseteq -least $x \in L$ with $f(x, gz) = f(\mathbf{id}_L \times g)(x, z) \sqsubseteq x$. Clearly, $f^\dagger gz$ is the same element.

To prove that the composition identity holds, suppose first that L, L' are models and $f : L \rightarrow L'$ and $g : L' \rightarrow L$ are α -monotonic for all $\alpha < \kappa$. Let $h = g \circ f : L \rightarrow L$ and $k = f \circ g : L' \rightarrow L'$. We want to show that $h^\dagger = g \circ k^\dagger$. Our argument uses the explicit construction of h^\dagger and k^\dagger .

Let $x_\alpha = \bigvee_{\beta < \alpha} y_\beta$ and $y_\alpha = h_\alpha x_\alpha$ for all $\alpha < \kappa$, so that $x_0 = \perp$, the least element of L . Similarly, let $x'_\alpha = \bigvee_{\beta < \alpha} y'_\beta$ and $y'_\alpha = k_\alpha x'_\alpha$ for all $\alpha < \kappa$. Thus, $x'_0 = \perp'$, the least element of L' . We know that the sequences $(y_\alpha)_{\alpha < \kappa}$ and $(y'_\alpha)_{\alpha < \kappa}$ are compatible, moreover, $h^\dagger = y = \bigvee_{\alpha < \kappa} y_\alpha$, $k^\dagger = y' = \bigvee_{\alpha < \kappa} y'_\alpha$. Also, $y|_\alpha = y_\alpha$ and $y'|_\alpha = y'_\alpha$, for all $\alpha < \kappa$.

We show by induction on α that $y_\alpha =_\alpha g y'_\alpha$. We will make use of the following lemma.

Lemma 4.3 Suppose that $x \in L$ and $x' \in L'$ with $x \sqsubseteq_\alpha g x'$ and $x' \sqsubseteq_\alpha f x$, where $\alpha < \kappa$. Let $y = h_\alpha x$ and $y' = k_\alpha x'$. Then $y =_\alpha g y'$ and $y' =_\alpha f y$.

First note that by $x \sqsubseteq_\alpha g x'$ and $x' \sqsubseteq_\alpha f x$, also $x \sqsubseteq_\alpha h x$ and thus $y = h_\alpha x$ exists. Similarly, $y' = k_\alpha x'$ also exists.

Now by the 1st clause of Lemma 3.4 $k y' \sqsubseteq_\alpha y'$, and since g is α -monotonic, also $h g y' = g k y' \sqsubseteq_\alpha g y'$. And since $x \sqsubseteq_\alpha g x'$ and $x' \sqsubseteq_\alpha y'$, also $x \sqsubseteq_\alpha g y'$. We conclude by the 2nd clause of Lemma 3.4 that $y \sqsubseteq_\alpha g y'$. Symmetrically, the same reasoning proves $y' \sqsubseteq_\alpha f y$.

Thus, $y \sqsubseteq_\alpha g y' \sqsubseteq_\alpha g f y = h y$. But by Theorem 3.1, it holds that $y =_\alpha h y$, so that $y =_\alpha g y'$. In a similar way, $y' =_\alpha f y$. This ends the proof of the lemma.

We now return to the main proof. In order to show that $y_\alpha =_\alpha g y'_\alpha$ and $y'_\alpha =_\alpha f y_\alpha$ hold for $\alpha = 0$, note that $x_0 = \perp \sqsubseteq_0 g \perp'_0 = g x'_0$, and symmetrically, $x'_0 = \perp' \sqsubseteq_0 f \perp = f x_0$. It follows by Lemma 4.3 that $y_0 =_0 g y'_0$ and $y'_0 =_0 f y_0$.

Suppose now that $\alpha > 0$ and our claim holds for all ordinals less than α . We distinguish two cases.

Case 1: $\alpha = \gamma + 1$ is a successor ordinal. Then, since the sequence $(y_\beta)_{\beta < \alpha}$ is compatible, by the induction hypothesis it holds that $x_\alpha = y_\gamma =_\gamma gy'_\gamma = gx'_\alpha$. But by the 3rd clause of Lemma 3.4, y_γ is a \sqsubseteq_α -least element of $[y_\gamma]_\gamma$, hence $x_\alpha \sqsubseteq_\alpha gx'_\alpha$. Symmetrically, $x'_\alpha \sqsubseteq_\alpha fx_\alpha$.

Case 2: α is a limit ordinal. Since the sequence $(y_\gamma)_{\gamma < \alpha}$ is compatible and hence increasing w.r.t. \leq , it holds that $x_\alpha = \bigvee_{\beta < \gamma < \alpha} y_\beta$ for all $\beta < \alpha$. By compatibility, $y_\beta =_\beta y_\gamma$ for all $\beta \leq \gamma$, so that by Ax3, $x_\alpha =_\beta y_\beta$ for all $\beta < \alpha$. Symmetrically, $x'_\alpha =_\beta y'_\beta$, and since g preserves the relation $=_\beta$, $gx'_\alpha =_\beta gy'_\beta$ for all $\beta < \alpha$. Also $y_\beta =_\beta gy'_\beta$ for all $\beta < \alpha$ by the induction hypothesis. This implies that $x_\alpha =_\beta y_\beta =_\beta gy'_\beta =_\beta gx'_\alpha$ for all $\beta < \alpha$. Since $(y_\gamma)_{\gamma < \kappa}$ is a compatible sequence, by Lemma 2.7, x_α is the \leq -least and a \sqsubseteq_α -least element of the set $(x_\alpha)_\alpha$. In particular, $x_\alpha \sqsubseteq_\alpha gx'_\alpha$. Symmetrically, $x'_\alpha \sqsubseteq_\alpha fx_\alpha$.

We have thus shown that in either case, $x_\alpha \sqsubseteq_\alpha gx'_\alpha$ and $x'_\alpha \sqsubseteq_\alpha fx_\alpha$. Thus, by Lemma 4.3, $y_\alpha =_\alpha gy'_\alpha$ and $y'_\alpha =_\alpha fy_\alpha$.

Now by $y = \bigvee_{\alpha < \kappa} y_\alpha$ and $y' = \bigvee_{\alpha < \kappa} y'_\alpha$ and since g is α -monotonic, it holds that $y|_\alpha = y_\alpha$, $y'|_\alpha = y'_\alpha$, and $y =_\alpha y_\alpha =_\alpha gy'_\alpha =_\alpha gy'$ for all $\alpha < \kappa$. Thus, by Ax2, $h^\dagger = y = gy' = gk^\dagger$. Symmetrically, $k^\dagger = fh^\dagger$.

In order to establish the composition identity in its general form (2), suppose now that L, L', L'' are models and $f : L \times L'' \rightarrow L'$ and $g : L' \times L'' \rightarrow L$ are α -monotonic for all $\alpha < \kappa$. We want to show that (2) holds. To this end, for every $z \in L''$, define $f_z : L \rightarrow L'$ and $g_z : L' \rightarrow L$ by $f_z x = f(x, z)$ and $g_z y = g(y, z)$ for all $x \in L$ and $y \in L'$. Then the functions f_z and g_z are also α -monotonic for all $\alpha < \kappa$. Moreover, since the parameter identity holds,

$$\begin{aligned} (f \circ \langle g, \pi_{L''}^{L' \times L''} \rangle)^\dagger z &= (f_z \circ g_z)^\dagger \\ (f \circ (\langle g \circ \langle f, \pi_{L''}^{L \times L''} \rangle)^\dagger) z &= f_z(g_z \circ f_z)^\dagger. \end{aligned}$$

Since by the above argument $(f_z \circ g_z)^\dagger = f_z(g_z \circ f_z)^\dagger$, hence

$$(f \circ \langle g, \pi_{L''}^{L' \times L''} \rangle)^\dagger z = (f \circ (\langle g \circ \langle f, \pi_{L''}^{L \times L''} \rangle)^\dagger) z.$$

Since this holds for all z , we established the composition identity.

Next we prove that the double dagger identity holds. First let $f : L \times L \rightarrow L$ be α -monotonic for all $\alpha < \kappa$, where L is a model. Since the fixed point identity holds,

$$\begin{aligned} f \circ \Delta_L \circ f^{\dagger\dagger} &= f \circ \langle f^{\dagger\dagger}, f^{\dagger\dagger} \rangle \\ &= f \circ \langle f^\dagger \circ f^{\dagger\dagger}, f^{\dagger\dagger} \rangle \\ &= f \circ \langle f^\dagger, \mathbf{id}_L \rangle \circ f^{\dagger\dagger} \\ &= f^\dagger \circ f^{\dagger\dagger} \\ &= f^{\dagger\dagger}. \end{aligned}$$

We conclude that $(f \circ \Delta_L)^\dagger \sqsubseteq f^{\dagger\dagger}$.

Suppose now that $g : \mathbf{1} \rightarrow L$ and

$$f \circ \Delta_L \circ g = f \circ \langle g, g \rangle \sqsubseteq g.$$

We want to show that $f^{\dagger\dagger} \sqsubseteq g$. But

$$f \circ \langle g, g \rangle = f \circ (\mathbf{id}_L \times g) \circ g$$

yielding

$$f \circ (\mathbf{id}_L \times g) \circ g \sqsubseteq g.$$

It follows that

$$(f \circ (\mathbf{id}_L \times g))^\dagger \sqsubseteq g.$$

Thus, by the parameter identity

$$f^\dagger \circ g \sqsubseteq g,$$

yielding $f^{\dagger\dagger} \sqsubseteq g$. Letting $g = (f \circ \Delta_L)^\dagger$, we conclude that $f^{\dagger\dagger} \sqsubseteq (f \circ \Delta_L)^\dagger$.

Now for the general case, let L and L' be models and suppose that $f : L \times L \times L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. Then f^\dagger and $f \circ (\Delta_L \times \mathbf{id}_{L'})$ are α -monotonic functions $L \times L' \rightarrow L$ for all $\alpha < \kappa$. Let $y \in L'$. We want to prove that $(f \circ (\Delta_L \times \mathbf{id}_{L'}))^\dagger y = f^{\dagger\dagger} y$. But using the notation introduced above, $(f \circ (\Delta_L \times \mathbf{id}_{L'}))^\dagger y = (f_y \circ \Delta_L)^\dagger$ and $f^{\dagger\dagger} y = ((f^\dagger)_y)^\dagger = (f_y)^\dagger$, moreover, $(f_y \circ \Delta_L)^\dagger = (f_y)^\dagger$ by the previous case.

We still need to show that the weak functorial implication holds. Actually we will show that a stronger property holds. We will make use of the following concept. Suppose that L and L' are models, $\alpha < \kappa$ and $h : L \rightarrow L'$. We say that h is strictly α -continuous if it is α -continuous, moreover, for each $x \in L$, $h(x|_\alpha) = (hx)|_\alpha$. Note that if $L' = L^n$, then the diagonal function $\Delta_L^n : L \rightarrow L^n$ is strictly α -continuous for all $\alpha < \kappa$, since if $x \in L$ and $\alpha < \kappa$, then $\Delta_L^n(x|_\alpha) = (x|_\alpha, \dots, x|_\alpha) = (x, \dots, x)|_\alpha = (\Delta_L^n x)|_\alpha$. Also note that Δ_L^n is continuous with respect to \leq and preserves the least element.

Claim. Suppose that L, L' are models, $f : L \rightarrow L$ and $g : L' \rightarrow L'$ are α -monotonic for all $\alpha < \kappa$ and $h : L \rightarrow L'$ is \leq -continuous and strictly α -continuous for all $\alpha < \kappa$ and preserves the least element. Suppose that $h \circ f = g \circ h$. Then $f^\dagger = h \circ g^\dagger$.

The proof of the claim relies on the explicit construction of f^\dagger and g^\dagger . We will make use of the following lemma.

Lemma 4.4 *Suppose that $x \in L$, $x' \in L'$ with $x \sqsubseteq_\alpha fx$ and $x' \sqsubseteq_\alpha gx'$, and let $y = f_\alpha x$, $y' = g_\alpha x'$. If $x' = hx$ then $y' = hy$.*

In order to prove this lemma, we follow the construction of $f_\alpha x$ and $g_\alpha x'$. Let $x_0 = x$, and for each successor ordinal $\lambda = \delta + 1$, define $x_\lambda = f x_\delta$. When λ is a limit ordinal, let $x_\lambda = \bigsqcup_\alpha \{x_\delta : \delta < \lambda\}$. Define the sequence $(x'_\lambda)_\lambda$ in a similar fashion starting with x' and using the function g . We prove by induction on λ that $hx_\lambda = x'_\lambda$.

When $\alpha = 0$, we have $hx_0 = hx = x' = x'_0$ by assumption. Suppose now that $\lambda > 0$ and our claim holds for all ordinals less than λ .

Let λ be a successor ordinal, say $\lambda = \delta + 1$. Then $hx_\lambda = hf x_\delta = gh x_\delta = gx'_\delta = x'_\lambda$, by the induction hypothesis. Suppose now that λ is a limit ordinal. Then $hx_\lambda = h(\bigsqcup_\alpha \{x_\delta : \delta < \lambda\}) =_\alpha \bigsqcup_\alpha \{hx_\delta : \delta < \lambda\} = \bigsqcup_\alpha \{x'_\delta : \delta < \lambda\} = x'_\lambda$ by the induction hypothesis and since h is α -continuous. But since $x_\lambda = x_\lambda|_\alpha$, $x'_\lambda = x'_\lambda|_\alpha$, $hx_\lambda =_\alpha x'_\lambda$ and h is strictly α -continuous, it follows that $hx_\lambda = x'_\lambda$.

Since there is some ordinal λ with $y = x_\lambda$ and $y'_\lambda = x'_\lambda$, the proof of the lemma is complete.

We now return to the proof of the claim. We know that f^\dagger can be constructed as follows. We define $x_\alpha, y_\alpha \in L$ for $\alpha < \kappa$ by $x_\alpha = \bigvee_{\beta < \alpha} y_\beta$ and $y_\alpha = f_\alpha x_\alpha$. Define $x'_\alpha, y'_\alpha \in L'$ in a similar way using the function g . Then $x_\alpha \sqsubseteq_\alpha f x_\alpha$ and $x'_\alpha \sqsubseteq_\alpha g x'_\alpha$ for all $\alpha < \kappa$, moreover, $f^\dagger = y = \bigvee_{\alpha < \kappa} y_\alpha$ and $g^\dagger = y' = \bigvee_{\alpha < \kappa} y'_\alpha$. Since $y_\alpha \leq y_\beta$ and $y'_\alpha \leq y'_\beta$ for all $\alpha < \beta < \kappa$, and since h is \leq -continuous, it follows by Ax2 that $g^\dagger = hf^\dagger$ if we can show that $y' =_\alpha hy$ for all $\alpha < \kappa$. But for all α , $y' =_\alpha hy$ holds iff $y'_\alpha =_\alpha hy_\alpha$, since $y =_\alpha y_\alpha$, $y' =_\alpha y'_\alpha$ and h is α -continuous. Thus, $hf^\dagger = g^\dagger$ holds if $y'_\alpha =_\alpha hy_\alpha$ for all $\alpha < \kappa$. Actually we will prove that $y'_\alpha = hy_\alpha$ for all $\alpha < \kappa$.

We prove by induction that $x'_\alpha = hx_\alpha$ and $y'_\alpha = hy_\alpha$ for all $\alpha < \kappa$. We have $x'_0 = \perp' = h\perp = hx_0$, since h preserves the least element, and thus $y'_0 = hy_0$ by Lemma 4.4. Suppose now that $\alpha > 0$ and that our claim holds for all $\beta < \alpha$. Now $x'_\alpha = \bigvee_{\beta < \alpha} y'_\beta = \bigvee_{\beta < \alpha} hy_\beta = h(\bigvee_{\beta < \alpha} y_\beta) = hx_\alpha$ by the induction hypothesis and since h is continuous. Moreover, $y'_\alpha = hy_\alpha$, again by Lemma 4.4.

Suppose now that L, L' are models and let $f : L \times L' \rightarrow L$ and $g : L^n \times L' \rightarrow L^n$ be α -monotonic for all $\alpha < \kappa$ such that $\Delta_L^n \circ f = g \circ (\Delta_L^n \times \mathbf{id}_{L'})$. Then for each fixed $y \in L'$, it holds that $\Delta_L^n \circ f_y = g_y \circ \Delta_L^n$. Thus, by the above claim, $\Delta_L^n(f_y)^\dagger = (g_y)^\dagger$, i.e., $\Delta_L^n f^\dagger y = g^\dagger y$. Since this holds for all y , we conclude that $\Delta_L^n \circ f^\dagger = g^\dagger$. \square

Corollary 4.5 *\mathbf{Mod}_c is an iteration category with a weak functorial dagger.*

By Remark 4.1, an identity involving the cartesian operations and dagger holds in the category of complete lattices and monotonic (or continuous) functions iff it holds in all iteration categories. Using this fact, Theorem 4.2 and Example 3.2, we obtain the following completeness result.

Corollary 4.6 (Completeness) *The following conditions are equivalent for an identity $t = t'$ between terms involving the cartesian operations and dagger:*

- $t = t'$ holds in \mathbf{Mod}_m .
- $t = t'$ holds in \mathbf{Mod}_c .
- $t = t'$ holds in iteration categories.

Proof. The fact that (i) implies (ii) is obvious. By Example 3.2, \mathbf{Mod}_c contains the category of complete lattices and continuous functions equipped with the least fixed point operation as external dagger. Hence, by Remark 4.1, any identity that holds in \mathbf{Mod}_c holds in iteration categories, proving that (ii) implies (iii). Finally, (iii) implies (i) by Theorem 4.2. \square

The same corollary may be derived from Theorem 4.2 and a result proved in [26] showing that every nontrivial iteration category having at least two morphisms $\mathbf{1} \rightarrow L$ for some object L satisfies exactly the identities of iteration theories.

Remark 4.7 *In [12], it is shown that least fixed points of monotonic functions over partial ordered sets give rise to iteration theories even if the least fixed points cannot be constructed. This result is not applicable here, since our functions are not necessarily monotonic w.r.t. \sqsubseteq . Moreover, we have also established weak functoriality that does not necessarily hold for the least fixed point operation.*

5 Cartesian closed categories and the abstraction identity

Following [18], in this section we define certain stronger versions of models that give rise to *cartesian closed categories* (ccc's) [1]. We establish the abstraction identity [5] that connects the fixed point operation to lambda abstraction.

For any objects L, L' in a ccc, we shall denote by $e_{L',L}$ an *evaluation morphism* $(L' \rightarrow L) \times L' \rightarrow L$. Thus, for any L, L', L'' and $f : L' \times L'' \rightarrow L$, there is a unique morphism $g : L'' \rightarrow (L' \rightarrow L)$ such that

$$f = e_{L',L} \circ (g \times \mathbf{id}_{L'}).$$

Below we will denote this unique morphism g by Λf .

We will consider ccc's equipped with an external dagger operation.

Remark 5.1 *An external dagger operation over a ccc satisfying the parameter identity may be internalized and is determined by a family of morphisms $(L \rightarrow L) \rightarrow L$, where L ranges over the objects. For the internal forms of the identities mentioned in this paper, we refer to [5].*

We recall a new axiom from [18].

Ax5. For all x_i, y_i with $x_i \sqsubseteq_\alpha y_i$ for all $i \in I$, where $\alpha < \kappa$ and I is an index set, it holds that $\bigvee \{x_i : i \in I\} \sqsubseteq_\alpha \bigvee \{y_i : i \in I\}$.

Note that in all models, Ax5 implies Ax4. Moreover, Ax5 trivially holds in all models when I is empty or a singleton set. Let \mathbf{Mod}'_m denote the full subcategory of \mathbf{Mod}_m spanned by those models satisfying Ax5. In [6], it is established in essence that \mathbf{Mod}'_m is a cartesian closed category.

Theorem 5.2 *The category \mathbf{Mod}'_m is cartesian closed.*

Proof. It is clear that the product of any family of models satisfying Ax5 also satisfies this axiom. Since \mathbf{Mod}_m is a cc, so is \mathbf{Mod}'_m .

As expected, for any models L, L' satisfying Ax5, the exponential object $(L' \rightarrow L)$ in \mathbf{Mod}'_m is the collection of all functions $f : L' \rightarrow L$ which are α -monotonic for all $\alpha < \kappa$. Since each function $L' \rightarrow L$ may be seen as an element of the product $\prod_{x \in L'} L$, equipped with the (pre)orderings \leq and \sqsubseteq_α , $\alpha < \kappa$,

defined pointwise, $(L' \rightarrow L)$ is a model satisfying *Ax5* provided that it is closed under the pointwise supremum operation \bigvee w.r.t \leq and the pointwise \bigwedge_γ operation, for all $\gamma < \kappa$.

To prove this, let G be a set of functions $L' \rightarrow L$ which are α -monotonic for all $\alpha < \kappa$. Define $f : L' \rightarrow L$ by $fx = \bigvee_{g \in G} gx$ for all $x \in L'$. Then f is also α -monotonic for all $\alpha < \kappa$. Indeed, if $x \sqsubseteq_\alpha y$ in L' , where $\alpha < \kappa$, then $gx \sqsubseteq_\alpha gy$ for all $g \in G$ as each $g \in G$ is α -monotonic. Thus, since *Ax5* holds in L , $fx = \bigvee_{g \in G} gx \sqsubseteq_\alpha \bigvee_{g \in G} gy = fy$.

Suppose now that $\bar{g} : L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. Let $\gamma < \kappa$, $G \subseteq (\bar{g}]_\gamma$ and $f = \bigwedge_\gamma G$. We want to prove that f is α -monotonic for all $\alpha < \kappa$. Suppose that $x \sqsubseteq_\alpha y$, where $\alpha < \kappa$. Since $G \subseteq (\bar{g}]_\gamma$, we have $\{gx : g \in G\} \subseteq (\bar{g}x]_\gamma$ and $\{gy : g \in G\} \subseteq (\bar{g}y]_\gamma$, so that $\bigwedge_\gamma \{gx : g \in G\}$ and $\bigwedge_\gamma \{gy : g \in G\}$ exist. Moreover, $fx = \bigwedge_\gamma \{gx : g \in G\}$ and $fy = \bigwedge_\gamma \{gy : g \in G\}$.

If $\gamma < \alpha$, then

$$fx = \bigwedge_\gamma \{gx : g \in G\} = \bigwedge_\gamma \{gy : g \in G\} = fy$$

since by $gx \sqsubseteq_\alpha gy$ and $\gamma < \alpha$ we have $gx =_\gamma gy$ for all $g \in G$.

Suppose now that $\gamma \geq \alpha$. Then by Lemma 2.6 (or Lemma 2.5),

$$\begin{aligned} (\bigwedge_\gamma \{gx : g \in G\})|_\alpha &= \bigwedge_\alpha \{gx : g \in G\} \\ &\sqsubseteq_\alpha \bigwedge_\alpha \{gy : g \in G\} \\ &= (\bigwedge_\gamma \{gy : g \in G\})|_\alpha, \end{aligned}$$

since $gx \sqsubseteq_\alpha gy$ for all $g \in G$. Hence $(fx)|_\alpha \sqsubseteq_\alpha (fy)|_\alpha$, so that $fx \sqsubseteq_\alpha fy$. (Note that the assumption that \bar{g} is α -monotonic is used in both cases when G is empty.) \square

The evaluation map $e_{L',L} : (L' \rightarrow L) \times L' \rightarrow L$ is the usual evaluation function $(f, x) \mapsto fx$. For each $\alpha < \kappa$, it is α -monotonic in its first argument due to the pointwise definition of the relation \sqsubseteq_α in $(L' \rightarrow L)$ and α -monotonic in its second argument since the functions in $(L' \rightarrow L)$ are α -monotonic. Finally, when $f : L' \times L'' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$, then for all $y \in L''$, define $(\Lambda f)y : L' \rightarrow L$ by $(\Lambda f)yx = f(x, y)$ for all $x \in L'$. Then $(\Lambda f)y$ is α -monotonic for all $\alpha < \kappa$ and $y \in L''$, as is Λf , since this holds for f . Indeed, if $\alpha < \kappa$, $y \in L''$ and $x \sqsubseteq_\alpha x'$ in L' , then

$$(\Lambda f)yx = f(x, y) \sqsubseteq_\alpha f(x', y) = (\Lambda f)yx'.$$

And if $y \sqsubseteq_\alpha y'$ in L'' then

$$(\Lambda f)y \sqsubseteq_\alpha (\Lambda f)y',$$

since for all x , $(\Lambda f)yx = f(x, y) \sqsubseteq_\alpha f(x, y') = (\Lambda f)y'x$. \square

We will now define a cartesian closed subcategory of \mathbf{Mod}_c . To this end, we introduce a new axiom. We say that *Ax6* holds in a model L if *Ax5* does and:

for all $\alpha < \kappa$, index set I and nonempty linearly ordered set (J, \leq) , and for all $x_{i,j} \in L$ where $i \in I$ and $j \in J$ such that $x_{i,j} \sqsubseteq_\alpha x_{i,k}$ whenever $j \leq k$, it holds:

$$\bigvee_{i \in I} \bigwedge_\alpha \{x_{i,j} : j \in J\} =_\alpha \bigwedge_\alpha \{ \bigvee_{i \in I} x_{i,j} : j \in J \}. \quad (3)$$

Remark 5.3 Note that the above condition in *Ax6* holds automatically when I is empty, for in that case both sides of (3) are equal to \perp . The axiom appears in [18] when J is the linearly ordered set of nonnegative integers.

Example 5.4 The standard model $V^{\mathbb{Z}}$ satisfies *Ax6*. See also [18]. Also, each complete lattice L viewed as a model as in Example 2.2 satisfies *Ax6*.

Let \mathbf{Mod}'_c denote the full subcategory of \mathbf{Mod}_c spanned by the models satisfying Ax6.

Theorem 5.5 *The category \mathbf{Mod}'_c is cartesian closed.*

Proof. First, if L, L' are models, then the collection of functions $L' \rightarrow L$ which are α -continuous for all $\alpha < \kappa$, equipped with the pointwise (pre)ordering relations \leq and \sqsubseteq_α , $\alpha < \kappa$, is also a model, giving rise to the exponential $(L' \rightarrow L)$. This is in part due to the observation that when G is a set of functions $g : L' \rightarrow L$ which are α -continuous for all $\alpha < \kappa$, then the pointwise supremum $\bigvee G$ is also α -continuous for all $\alpha < \kappa$. The proof of this fact uses Ax6. Indeed, let (I, \leq) be a nonempty linearly ordered set and $x_i \in L'$ for all $i \in I$ such that $x_i \sqsubseteq_\alpha x_j$ whenever $i \leq j$ in I . Then, using Ax5 and Ax6 and the assumption that each $g \in G$ is α -continuous,

$$\begin{aligned} (\bigvee_\alpha G)(\bigsqcup_\alpha \{x_i : i \in I\}) &= \bigvee_{g \in G} g(\bigsqcup_\alpha \{x_i : i \in I\}) \\ &=_\alpha \bigvee_{g \in G} \bigsqcup_\alpha \{gx_i : i \in I\} \\ &=_\alpha \bigsqcup_\alpha \bigvee_{g \in G} \{gx_i : i \in I\} \\ &= \bigsqcup_\alpha \{(\bigvee G)x_i : i \in I\}. \end{aligned}$$

(When G is empty, both sides are equal to \perp .)

Suppose now that $\bar{g} : L' \rightarrow L$ is α -continuous for all $\alpha < \kappa$ and $G \subseteq (\bar{g}]_\gamma$, where $\gamma < \kappa$ such that each $g \in G$ is α -continuous for all $\alpha < \kappa$. We still need to prove that $f = \bigsqcup_\gamma G$ is α -continuous for all $\alpha < \kappa$.

Let (I, \leq) be a nonempty linearly ordered set and $x_i \in L'$ for all $i \in I$ such that $x_i \sqsubseteq_\alpha x_j$ whenever $i \leq j$ in I . Then $gx_i \sqsubseteq_\alpha gx_j$ for all $g \in G$ and $i \leq j$ in I . Moreover, since $G \subseteq (\bar{g}]_\gamma$, $\{gx_i : g \in G\} \subseteq (\bar{g}x_i]_\gamma$ for all $i \in I$. Hence, $\bigsqcup_\gamma \{gx_i : g \in G\} \in (\bar{g}x_i]_\gamma$ for all $i \in I$.

Let $f = \bigsqcup_\alpha G$. We want to prove that $f(\bigsqcup_\alpha \{x_i : i \in I\}) =_\alpha \bigsqcup_\alpha \{fx_i : i \in I\}$.

First let $\gamma < \alpha$. Then

$$\begin{aligned} f(\bigsqcup_\alpha \{x_i : i \in I\}) &= \bigsqcup_\gamma \{g(\bigsqcup_\alpha \{x_i : i \in I\}) : g \in G\} \\ &= \bigsqcup_\gamma \{\bigsqcup_\alpha \{gx_i : i \in I\} : g \in G\} \\ &= \bigsqcup_\gamma \{gx_i : i \in I, g \in G\} \end{aligned}$$

where the third equality uses Lemma 2.6 and the second equality is due to the fact that since $g(\bigsqcup_\alpha \{x_i : i \in I\}) =_\alpha \bigsqcup_\alpha \{gx_i : i \in I\}$ by α -continuity, also $g(\bigsqcup_\alpha \{x_i : i \in I\}) =_\gamma \bigsqcup_\alpha \{gx_i : i \in I\}$ by $\gamma < \alpha$ for all $g \in G$. On the other hand,

$$\begin{aligned} \bigsqcup_\alpha \{fx_i : i \in I\} &= \bigsqcup_\alpha \{\bigsqcup_\gamma \{gx_i : g \in G\} : i \in I\} \\ &= \bigsqcup_\alpha \{\bigsqcup_\gamma \{gx_i : g \in G, i \in I\}\} \\ &= (\bigsqcup_\gamma \{gx_i : g \in G, i \in I\})|_\alpha \end{aligned}$$

as $\bigsqcup_\gamma \{gx_i : g \in G\} = \bigsqcup_\gamma \{gx_j : g \in G\}$ for all $i = j$. (Hint: since $\gamma < \alpha$, $x_i =_\gamma x_j$ and $gx_i =_\gamma gx_j$ for all $i, j \in I$ and $g \in G$.) But by Lemma 2.5,

$$(\bigsqcup_\gamma \{gx_i : g \in G, i \in I\})|_\alpha = \bigsqcup_\gamma \{gx_i : g \in G, i \in I\},$$

so that

$$f(\bigsqcup_{\alpha} \{x_i : i \in I\}) = (\bigsqcup_{\alpha} \{fx_i : i \in I\})|_{\alpha},$$

ie.,

$$f(\bigsqcup_{\alpha} \{x_i : i \in I\}) =_{\alpha} \bigsqcup_{\alpha} \{fx_i : i \in I\},$$

Next, suppose that $\gamma \geq \alpha$. Then using Lemma 2.6 in the second, fourth and fifth lines and α -continuity in the third,

$$\begin{aligned} (f(\bigsqcup_{\alpha} \{x_i : i \in I\}))|_{\alpha} &= (\bigsqcup_{\gamma} \{g(\bigsqcup_{\alpha} \{x_i : i \in I\}) : g \in G\})|_{\alpha} \\ &= \bigsqcup_{\alpha} \{g(\bigsqcup_{\alpha} \{x_i : i \in I\}) : g \in G\} \\ &= \bigsqcup_{\alpha} \{\bigsqcup_{\alpha} \{gx_i : i \in I\} : g \in G\} \\ &= \bigsqcup_{\alpha} \{\bigsqcup_{\alpha} \{\bigsqcup_{\gamma} \{gx_i : i \in I\}\} : g \in G\} \\ &= \bigsqcup_{\alpha} \{\bigsqcup_{\gamma} \{gx_i : i \in I\} : g \in G\} \\ &= \bigsqcup_{\alpha} \{fx_i : i \in I\}, \end{aligned}$$

so that

$$(f(\bigsqcup_{\alpha} \{x_i : i \in I\}))|_{\alpha} = \bigsqcup_{\alpha} \{fx_i : i \in I\}.$$

Thus, $f(\bigsqcup_{\alpha} \{x_i : i \in I\}) =_{\alpha} \bigsqcup_{\alpha} \{fx_i : i \in I\}$ again. (The assumption that \bar{g} is α -continuous has been used implicitly when G is empty.)

Now, for any pair of models L, L' and any $\alpha < \kappa$, $\bar{f} : L' \rightarrow L$ in \mathbf{Mod}'_c and for all $F \subseteq (L' \rightarrow L) \subseteq (\bar{f})_{\alpha}$, the evaluation function $e = e_{L', L}$ satisfies $e(\bigsqcup_{\alpha} F, x) = \bigsqcup_{\alpha} \{e(f, x) : f \in F\}$ since $\bigsqcup_{\alpha} F$ is formed pointwise. And if (J, \leq) is a nonempty linearly ordered set and $x_j \in L'$ for all $j \in J$ such that $x_j \sqsubseteq_{\alpha} x_k$ whenever $j \leq k$ in J , and if $f \in (L' \rightarrow L)$, then $e(f, \bigsqcup_{\alpha} \{x_j : j \in J\}) =_{\alpha} \bigsqcup_{\alpha} \{e(f, x_j) : j \in J\}$ since f is α -continuous. Finally, if $f : L' \times L'' \rightarrow L$ is α -continuous for all $\alpha < \kappa$, where L, L', L'' are models, then $(\Lambda f)y : L' \rightarrow L$ is also α -continuous for all $y \in L''$ and $\alpha < \kappa$ as is $\Lambda f : L'' \rightarrow (L' \rightarrow L)$. \square

The abstraction identity was introduced in [5] in order to connect the Conway structure to exponentials in those ccc's which are Conway categories.

Abstraction identity

$$\Lambda(f^{\dagger}) = (\Lambda g)^{\dagger}$$

where $f : L \times L' \times L'' \rightarrow L$ and

$$g = f \circ ((\langle e_{L', L}, \pi_{L'}^{(L' \rightarrow L) \times L'} \rangle \circ \langle \pi_{(L' \rightarrow L)}^{L' \times (L' \rightarrow L)}, \pi_{L'}^{L' \times (L' \rightarrow L)} \rangle) \times \mathbf{id}_{L''}) : L' \times (L' \rightarrow L) \times L'' \rightarrow L,$$

so that

$$\Lambda g : (L' \rightarrow L) \times L'' \rightarrow (L' \rightarrow L).$$

Below we will establish the abstraction identity in \mathbf{Mod}'_m and \mathbf{Mod}'_c . We will make use of some further results including the fixed point induction rule of Theorem 5.7.

Lemma 5.6 *Suppose that L is a model and $f : L \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. Let $\alpha < \kappa$ and $z \in L$ with $fz \sqsubseteq_{\alpha} z$. Then either there is some $\beta \leq \alpha$ with $f^{\dagger} \sqsubseteq_{\beta} z$, or $f^{\dagger} =_{\alpha} z$.*

Proof. Define $x_\alpha = \bigvee_{\gamma < \alpha} y_\gamma$ and $y_\alpha = f_\alpha x_\alpha$ for all ordinals $\alpha < \kappa$. We know that $f^\dagger|_\alpha = y_\alpha$ for all α . Thus, for all $\alpha < \kappa$ and $z \in L$ with $f(z) \sqsubseteq_\alpha z$, we have $f^\dagger \sqsubseteq_\alpha z$ iff $y_\alpha \sqsubseteq_\alpha z$ and $f^\dagger =_\alpha z$ iff $y_\alpha =_\alpha z$.

We prove by induction on $\alpha < \kappa$ that either there is some $\beta \leq \alpha$ with $y_\beta \sqsubset_\beta z$, or $y_\alpha =_\alpha z$. When $\alpha = 0$ this is clear, since $x_0 = \perp \sqsubseteq_0 z$, hence by Lemma 3.4, $y_0 \sqsubseteq_0 z$.

Suppose that $\alpha > 0$ and our claim holds for all ordinals less than α . If there is some $\gamma < \alpha$ with $y_\gamma \sqsubset_\gamma z$ we are done. So without loss of generality we may assume that $y_\gamma =_\gamma z$ for all $\gamma < \alpha$. There are two cases.

Suppose first that α is a successor ordinal, say $\alpha = \gamma + 1$. Then $z \in [y_\gamma]_\gamma$, hence by the third clause of Lemma 3.4, $y_\gamma \sqsubseteq_\alpha z$. But $x_\alpha = y_\gamma$, thus $x_\alpha \sqsubseteq_\alpha z$, and since $f(z) \sqsubseteq_\alpha z$, we conclude that $y_\alpha \sqsubseteq_\alpha z$ by the second clause of Lemma 3.4.

Suppose now that α is a limit ordinal. Since $y_\gamma =_\gamma z$ for all $\gamma < \alpha$, $x_\alpha = \bigvee_{\gamma < \alpha} y_\gamma \sqsubseteq_\alpha z$ by Lemma 2.7. Since $f(z) \sqsubseteq_\alpha z$, it follows by Lemma 3.4 again that $y_\alpha \sqsubseteq_\alpha z$. \square

If L and L' are models, we let $L^{L'}$ denote the model of all functions $L' \rightarrow L$ which is isomorphic to the L' -fold direct product of L with itself. By Theorem 3.1 and Lemma 5.6 we have:

Corollary 5.7 *Suppose that L and L' are models and $f : L \times L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$. If $g : L \times L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$ such that $f \circ \langle g, \text{id}_{L'} \rangle \sqsubseteq g$ in the model $L^{L'}$, then $f^\dagger \sqsubseteq g$.*

Proof. Suppose that $g : L' \rightarrow L$ is α -monotonic for all $\alpha < \kappa$ with $f \circ \langle g, \text{id}_{L'} \rangle \sqsubseteq g$. If equality holds, then $f(gy, y) = y$ for all $y \in L'$, hence, $f^\dagger y \sqsubseteq gy$ for all $y \in L'$ by Theorem 3.1. Thus for each y , either $f^\dagger y = gy$, or there is some $\alpha_y < \kappa$ with $f^\dagger y \sqsubset_{\alpha_y} gy$. When $f^\dagger y = gy$, define $\alpha_y = \kappa$, and then let α be the least ordinal in the set $\{\alpha_y : y \in L'\}$. If $\alpha = \kappa$ then $f^\dagger = g$, otherwise $f^\dagger \sqsubset_\alpha g$. In either case, $f^\dagger \sqsubseteq g$.

Suppose now that $f \circ \langle g, \text{id}_{L'} \rangle \sqsubset g$. Then by Remark 2.15, there is some $\alpha < \kappa$ such that for all $y \in L'$ it holds that $f(gy, y) \sqsubseteq_\alpha y$, and there is some $y_0 \in L'$ with $f(gy_0, y_0) \sqsubset_\alpha y_0$. Now by Lemma 5.6 (applied to f_y), for each y either $f^\dagger y =_\alpha gy$, or there is an ordinal $\beta_y \leq \alpha$ with $f^\dagger y \sqsubset_{\beta_y} gy$. Let $\beta = \min\{\alpha, \beta_y : y \in L'\}$. Then $f^\dagger \sqsubset_\beta g$, hence $f^\dagger \sqsubset g$. \square

Theorem 5.8 *The abstraction identity holds in the categories \mathbf{Mod}'_m and \mathbf{Mod}'_c .*

Proof. We prove this result only for morphisms $f : L \times L' \rightarrow L$ (i.e., when L'' is a singleton). In that case the assertion becomes $f^\dagger = (\Lambda g)^\dagger$, where

$$g = f \circ \langle e_{L', L}, \pi_{L'}^{(L' \rightarrow L) \times L'} \rangle \circ \langle \pi_{(L' \rightarrow L)}^{L' \times (L' \rightarrow L)}, \pi_{L'}^{L' \times (L' \rightarrow L)} \rangle : (L' \rightarrow L) \rightarrow (L' \rightarrow L),$$

so that $\Lambda g : (L' \rightarrow L) \rightarrow (L' \rightarrow L)$.

Notice that for all $f \in (L' \rightarrow L)$ and $y \in L'$, $(\Lambda g)hy = f(hy, y)$, hence

$$(\Lambda g)h = f \circ \langle h, \text{id}_{L'} \rangle : L' \rightarrow L.$$

Thus we have

$$(\Lambda g)f^\dagger = f \circ \langle f^\dagger, \text{id}_{L'} \rangle = f^\dagger,$$

since the fixed point identity holds. Suppose that $h \in (L' \rightarrow L)$ with $(\Lambda g)h \sqsubseteq h$. Then $f \circ \langle h, \text{id}_{L'} \rangle \sqsubseteq h$, hence $f^\dagger \sqsubseteq h$ by Corollary 5.7.

We have proved that f^\dagger is the least fixed point of (Λg) with respect to the ordering \sqsubseteq . Since $(\Lambda g)^\dagger$ is also a least fixed point, we conclude that $(\Lambda g)^\dagger = f^\dagger$. \square

6 Some variants of the categories

Several subcategories of \mathbf{Mod}_m and \mathbf{Mod}_c were introduced in [6, 18, 19] in connection with logic programming and boolean grammars. In this section we mention some of them and establish that

they are also cc's and/or ccc's equipped with an external dagger operation satisfying the identities of iteration categories and, if applicable, the abstraction identity.

Suppose that L is a model. We say that L is a strong model if it satisfies the following two axioms:

Ax7. For all $x, y \in L$ and $\alpha < \kappa$, if $x \leq y$ and $x =_\beta y$ for all $\beta < \alpha$, then $x \sqsubseteq_\alpha y$.

Ax8. For all $x, y \in L$, if $x \leq y$ then $x|_\alpha \leq y|_\alpha$ for all $\alpha < \kappa$.

Example 6.1 *Again, the standard model V^Z satisfies these axioms as does every complete lattice as a model, cf. 2.2.*

Remark 6.2 *In any model L , it holds that $x = \bigvee_{\alpha < \kappa} x|_\alpha$, for all $x \in X$. It follows that if $x, y \in L$ with $x|_\alpha \leq y|_\alpha$ for all $\alpha < \kappa$, then $x \leq y$. If Ax8 holds, then the converse is also valid, so that for all $x, y \in L$, $x \leq y$ iff $x|_\alpha \leq y|_\alpha$ for all $\alpha < \kappa$.*

As noted in [18], if a model L satisfies Ax7, then the relation \leq is included in the relation \sqsubseteq . Thus, in such models L , the greatest elements w.r.t. \leq and \sqsubseteq coincide.

Proposition 6.3 *Suppose that L_i is a model for all $i \in I$ and let $L = \prod_{i \in I} L_i$. Then L satisfies Ax7 iff each L_i does. Similarly, L satisfies Ax8 iff each L_i does. Thus, L is a strong model iff each L_i is a strong model.*

Proof. Immediate from the pointwise definition of the order relations in L . □

Proposition 6.4 *Suppose that L and L' are models. If L satisfies Ax5, Ax7 and Ax8, then so does $(L' \rightarrow L)$ in \mathbf{Mod}_m . Thus, if L and L' are strong models satisfying Ax5, then so is $(L' \rightarrow L)$ in \mathbf{Mod}_m .*

Proof. This follows from the previous proposition and the fact that any product of models satisfying Ax5 also satisfies this axiom, since $(L' \rightarrow L)$ can be embedded in $\prod_{x \in L'} L$. □

Similarly, we have:

Proposition 6.5 *Suppose that L and L' are models. If L satisfies Ax6, Ax7 and Ax8, then so does $(L' \rightarrow L)$ in \mathbf{Mod}_c . Thus, if L and L' are strong models satisfying Ax6, then so is $(L' \rightarrow L)$ in \mathbf{Mod}_c .*

Let \mathbf{SMod}_m and \mathbf{SMod}_c denote the full subcategories of \mathbf{Mod}_m and \mathbf{Mod}_c determined by the strong models. Similarly, let \mathbf{SMod}'_m and \mathbf{SMod}'_c denote the full subcategories of \mathbf{Mod}'_m and \mathbf{Mod}'_c determined by the strong models.

Corollary 6.6 *\mathbf{SMod}_m and \mathbf{SMod}_c , equipped with the external dagger operations inherited from \mathbf{Mod}_m and \mathbf{Mod}_c , respectively, are iteration categories. \mathbf{SMod}'_m and \mathbf{SMod}'_c are ccc's satisfying the identities of iteration categories and the abstraction identity.*

Ax1 and Ax2 are self-dual but Ax3 and Ax4 are not. Their duals are the following axioms.

Ax3d. For every $\alpha < \kappa$, $x \in L$ and $X \subseteq (x)_\alpha$ there exists some $z \in (x)_\alpha$ with the following properties:

- $z \sqsubseteq_\alpha X$,
- for all $y \in (x)_\alpha$, if $y \sqsubseteq_\alpha X$ then $y \sqsubseteq_\alpha z$ and $y \leq z$.

Ax4d. For every $\alpha < \kappa$ and nonempty $X \subseteq L$ and $y \in L$, if $X =_\alpha y$, then $\bigwedge X =_\alpha y$.

The element z is unique in Ax3d and we denote it by $\bigcap_\alpha X$. In particular, let $x|^\alpha = \bigcap_\alpha \{x\}$ for all $x \in X$ and $\alpha < \kappa$.

Regarding Ax7 and Ax8, Ax7 is self-dual but Ax8 is not. The dual of Ax8 is:

Ax8d. For all $x, y \in L$, if $x \leq y$ then $x|^\alpha \leq y|^\alpha$ for all $\alpha < \kappa$.

We say that a model L is a symmetric model if it also satisfies $Ax3d$ and $Ax4d$. A symmetric strong model is a symmetric model which is a strong model satisfying $Ax8d$. Let \mathbf{SymMod}_m and \mathbf{SymMod}_c denote the full subcategories of \mathbf{Mod}_m and \mathbf{Mod}_c spanned by the symmetric models. Similarly, let $\mathbf{SymSMod}_m$ and $\mathbf{SymSMod}_c$ denote the full subcategories of \mathbf{Mod}_m and \mathbf{Mod}_c determined by the symmetric strong models.

Corollary 6.7 *\mathbf{SymMod}_m , \mathbf{SymMod}_c , $\mathbf{SymSMod}_m$ and $\mathbf{SymSMod}_c$, equipped with the external dagger operation inherited from \mathbf{Mod}_m and \mathbf{Mod}_c , are cc's satisfying all identities of iteration categories.*

Since the model obtained from a complete lattice as in Example 2.2 is a symmetric strong model, it follows that an identity involving the cartesian operations and dagger holds in any of the categories \mathbf{SMod}'_m , \mathbf{SMod}'_c , \mathbf{SymMod}_m , \mathbf{SymMod}_c , $\mathbf{SymSMod}_m$ and $\mathbf{SymSMod}_c$ iff it holds in iteration categories.

One may also define a cartesian closed categories of symmetric models and symmetric strong models as subcategories of \mathbf{Mod}'_m . These are ccc's and iteration categories and satisfy the abstraction identity. We skip the details.

Similarly to Corollary 4.6, each of the above categories satisfies an identity involving the cartesian operations and dagger iff the identity holds in all iteration categories.

7 Conclusion

We introduced several cartesian and cartesian closed categories of stratified complete lattices and equipped them with the stratified least fixed point operation as external dagger. We proved that all identities of iteration categories hold in these categories. And since each of these categories 'extends' the category of complete lattices and monotonic or continuous functions equipped with the least fixed point operation as dagger, we concluded that an identity involving the cartesian operations and dagger holds in any of these categories iff it holds in all iteration categories. When the category is cartesian closed, we also established the abstraction identity. One may thus perform symbolic computations over these categories using *the* standard equational properties of fixed point operations. This has significance for modular logic programming, see eg. [22].

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